Journal of Approximation Theory **99**, 369–408 (1999)
Article ID jath.1999.3333, available online at http://www.idealibrary.com on IDEAL®

On the Convergence of Bounded J-Fractions on the Resolvent Set of the Corresponding Second Order Difference Operator

Bernhard Beckermann

Laboratoire d'Analyse Numérique et d'Optimisation, UFR IEEA-M3, Université de Lille 1, F-59655 Villeneuve d'Ascq Cedex, France
E-mail: bbecker@ano.univ-lille1.fr

Communicated by Guillermo López Lagomasino

Received October 7, 1997; accepted in revised form November 5, 1998

We study connections between continued fractions of type J and spectral properties of second order difference operators with complex coefficients. It is known that the convergents of a bounded J-fraction are diagonal Padé approximants of the Weyl function of the corresponding difference operator and that a bounded J-fraction converges uniformly to the Weyl function in some neighborhood of infinity. In this paper we establish convergence in capacity in the unbounded connected component of the resolvent set of the difference operator and specify the rate of convergence. Furthermore, we show that the absence of poles of Padé approximants in some subdomain implies already local uniform convergence. This enables us to verify the Baker–Gammel–Wills conjecture for a subclass of Weyl functions. For establishing these convergence results, we study the ratio and the nth root asymptotic behavior of Padé denominators of bounded J-fractions and give relations with the Green function of the unbounded connected component of the resolvent set. In addition, we show that the number of "spurious" Padé poles in this set may be bounded.

Key Words: difference operator; Padé approximation; Weyl function; convergence of J-fractions; Baker–Gammel–Wills conjecture.

1. INTRODUCTION

We consider the convergence of the sequence of convergents of continued fractions of a particular form, so-called *bounded J-fractions* [24, Chap. V.26]

$$\frac{1}{|z-b_0}+\frac{-a_0^2|}{|z-b_1}+\frac{-a_1^2|}{|z-b_2}+\frac{-a_2^2|}{|z-b_3}+\cdots, \tag{1}$$

where a_j, b_j are some complex numbers, and $a_j \neq 0$ for all j, so that the continued fraction is not terminating. Our main restriction throughout

0021-9045/99 $30.00

this paper is that these coefficients are *uniformly bounded.* We recall from [24, Chap. VIII] that the nth convergent of the continued fraction (1)

$$\pi_n(z) := \frac{1}{|z-b_0} + \frac{-a_0^2|}{|z-b_1} + \frac{-a_1^2|}{|z-b_2} + \frac{-a_2^2|}{|z-b_3} + \cdots + \frac{-a_{n-2}^2|}{|z-b_{n-1}} \tag{2}$$

may be rewritten as $\pi_n = p_n/q_n$ ($n = 1, 2, \ldots$), with p_n a polynomial of degree at most $n-1$, and q_n a polynomial of degree n. Also, the sequences $(p_n(z))_{n\geqslant 0}$, $(q_n(z))_{n\geqslant 0}$ can be obtained as solutions of the three term recurrence relation

$$z \cdot y_n = a_{n-1} \cdot y_{n-1} + b_n \cdot y_n + a_n \cdot y_{n+1}, \qquad n \geqslant 0, \tag{3}$$

together with the initializations

$$q_0(z) = 1, \qquad q_{-1}(z) = 0, \qquad p_0(z) = 0, \qquad p_{-1}(z) = -1, \tag{4}$$

where $a_{-1} = 1$. In [24, Theorem V.26.3], Wall establishes uniform convergence of the sequence of convergents $(\pi_n)_{n\geqslant 0}$ in some neighborhood of infinity to some function ϕ which is therefore analytic in this neighborhood. Also, it is well known (see, e.g., [24, Chap. VIII]) that

$$r_n(z) := q_n(z) \cdot \phi(z) - p_n(z) = \frac{d_0}{z^{n+1}} + \frac{d_1}{z^{n+2}} + \cdots, \tag{5}$$

with r_n being referred to as the *residual.* Thus, the rational function $\pi_n = p_n/q_n$, is the (diagonal) Padé approximant of ϕ order n (at infinity). We recall that, by a change of variables, the nth Padé approximant of at infinity becomes the ordinary $[n \mid n]$ Padé approximant of $\phi(1/z)$ (at zero) [3].

Conversely, as a starting point we may suppose that ϕ is some *regular* power series around infinity

$$\phi(z) = \sum_{j=0}^{\infty} \frac{c_j}{z^{j+1}}, \tag{6}$$

i.e., the Padé approximants $\pi_n = p_n/q_n$ of any order n exist and are pairwise distinct. Then it is known (see, e.g., [16, Proposition 4.2]) that the numerators and denominators of any three consecutive Padé approximants are related by a recurrence relation of the form (3), (4), and therefore π_n, $n \geqslant 1$, has the continued fraction representation (2) with coefficients a_j, b_j independent of n. In addition, the expansion around infinity of ϕ and of the continued J-fraction (1) coincide.

A third approach to the continued fraction (1) is given by exploiting the connection between (3) and some second order difference operators. Consider the infinite tridiagonal matrix

$$\mathscr{A} := \begin{pmatrix} b_0 & a_0 & 0 & \cdots & \cdots \\ a_0 & b_0 & a_1 & 0 & \cdots \\ 0 & a_1 & b_2 & a_2 & \cdots \\ \ddots & \ddots & \ddots & \ddots & \ddots \end{pmatrix}. \tag{7}$$

We denote by ℓ^2 the Hilbert space of complex quadratic summable sequences and by $(e_n)_{n \geqslant 0}$ its usual orthonormal basis. In the sequel we will associate with the matrix $\mathscr{A}$ a linear operator A in ℓ^2 in the following way: first we define the operator by the usual matrix product on the set of finite linear combinations of the basis elements $e_0, e_1, ...,$ and then we take its closure. Since the entries a_j, b_j are supposed to be uniformly bounded, the resulting operator A is defined on the whole set ℓ^2 and bounded. Notice that A is self-adjoint if and only if a_j, b_j are real for all j. Let $\sigma(A)$ be the spectrum of the operator, $\Omega(A) = \mathbb{C} \backslash \sigma(A)$ its resolvent set, i.e., the set of all numbers $z \in \mathbb{C}$ so that $(z \cdot I - A)$ is boundedly invertible in ℓ^2. The operator-valued function $R(z) := (z \cdot I - A)^{-1}$, analytic in $\Omega(A)$, is called resolvent of A. Following Berezanskii (see [9]), we call

$$\phi(z) := (R(z)\, e_0, e_0) = \sum_{j=0}^{\infty} \frac{(A^j e_0, e_0)}{z^{j+1}} \tag{8}$$

the *Weyl function* of A, analytic in the resolvent set of A, and thus in particular in the neighborhood $|z| > \|A\|$ around infinity. Define the sequences of polynomials $(q_n)_{n \geqslant 0}$ and $(p_n)_{n \geqslant 0}$ as solutions of (3) together with the initial conditions (4). Then, for any $\geqslant 1$, the rational function p_n/q_n turns out to be the nth Padé approximant of the power series (8) (see [13]). Note that the spectral equation $Ay = z \cdot y$ reduces to the difference equation (3). In fact, Padé approximants of a Weyl function are useful tools in the spectral theory of second order difference operators, which again have applications to non-linear discrete dynamical systems (see [2, 8, 9, 16] and the references therein). We also mention that, instead of (7), we may also allow arbitrary complex entries different from zero on both the superdiagonal and the subdiagonal, corresponding to a different scaling of the Padé numerators p_n, and denominators q_n. However, we will restrict our attention in the present paper to the subclass (7) of second order difference operators since, as shown in [8, Theorem 2.3], the above scaling leads to a maximal resolvent set.

All the subsequent considerations are in the extended complex plane $\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$, equipped with the chordal metric. We say that some property is valid locally uniformly in some open set $D \subset \overline{\mathbb{C}}$ if it holds uniformly on closed subsets of D. For instance, $\limsup_{n\to\infty} f_n(z) \leqslant f(z)$ holds locally uniformly in D if for every closed $F \subset D$ and for every $\varepsilon > 0$ there exists an $N(\varepsilon, F)$ so that $f_n(z) \leqslant f(z) + \varepsilon$ for all $z \in F$ and for all $n \geqslant N(\varepsilon, F)$. Also, in what follows we fix A and write shorter $\Omega := \Omega(A)$, with Ω_0 being the outer domain, i.e., the connected component of Ω containing infinity. Furthermore, the (generalized) Green function of Ω_0 with a pole at infinity will be denoted by g_{Ω_0}. Notice that for the particular case of self-adjoint operators A there holds $\Omega = \Omega_0$, and $\sigma(A)$ becomes a compact subset of the real line. However, there exist bounded operators A with matrix representation (7) having a resolvent set consisting of several connected components, i.e., $\Omega \neq \Omega_0$ (see Example 5.2 below).

In order to motivate our results presented below, let us shortly recall some properties of Padé approximants for the particular case of real bounded recurrence coefficients (i.e., A is a bounded self-adjoint operator). By the spectral theorem, the Weyl function (8) may be represented as a Markov function, with the corresponding measure being the spectral measure of A, supported on $\sigma(A)$. The Markov theorem provides local uniform convergence of the sequence $(\pi_n)_{n\geqslant 0}$ of Padé approximants to the Weyl function ϕ in $\mathbb{C}\backslash\mathcal{S}$, where $\mathcal{S}$ is the convex hull of $\sigma(A)$ (see, e.g., [16, Chap. 2.6]). In particular, all Padé poles lie in $\mathcal{S}$. On the other hand, if $\sigma(A) \neq \mathcal{S}$ then an infinite number of Padé approximants may have so-called *spurious poles* in Ω, namely in the gaps of the spectrum. Thus in general we do not have local uniform convergence of the whole sequence $(\pi_n)_{n\geqslant 0}$ in the whole resolvent set. Let us also recall a result of Widom [25] who showed that, for any closed set $F \subset \Omega(A)$, the number of zeros of q_n in F may depend on n, but is bounded in n. Finally, from [20, Chap. 1.1] or [16, Chap. 2.8] we obtain information about the growth of Padé denominators on and outside the spectrum. In particular, from [20, Theorem 1.1.4] one may deduce convergence of the sequence of Padé approximants in capacity in the resolvent set.

For arbitrary bounded recurrence coefficients a_j, b_j, we show in Theorem 3.1 of Section 3 that there is convergence in capacity of the whole sequence of Padé approximants in the outer domain Ω_0 of the resolvent set. The restriction to the outer domain is natural, since the Weyl function ϕ is approximated in terms of its Laurent series at infinity; moreover, according to the special case of self-adjoint operators we may not expect that a sharper form of convergence is valid in the whole outer domain Ω_0.

For obtaining convergence in capacity and for specifying the rate of convergence (see Proposition 3.2) we essentially need three elements: first we show in Proposition 2.1 that the result of Widom on the number of poles in the outer component of the resolvent set is also true in our general setting. As a second element we examine in Proposition 2.2 the asymptotic behavior of the ratio of two consecutive denominators. Finally, in the remaining part of Section 2 we study the two functions $g_{\inf}$ and $g_{\sup}$, defined on Ω by

$$g_{\inf}(z) := \liminf_{n\to\infty} \log[\,|q_n(z)|^2 + |a_n \cdot q_{n+1}(z)|^2]^{1/2(n+1)}, \tag{9}$$

$$g_{\sup}(z) := \limsup_{n\to\infty} \log |q_n(z)|^{1/n}. \tag{10}$$

For self-adjoint A, properties of these functions are well-established; see for instance the monograph [20] of Stahl and Totik on general orthogonal polynomials. In the general case, some asymptotic properties for $g_{\sup}$ have been given by Aptekarev, *et al.* [2, Corollary 3]; a function closely related to $g_{\inf}$ was studied by Kaliaguine and the present author [8, Theorem 3.6]. Beside several other characterizations, we will show in Theorem 2.5 that $g_{\inf}$, $g_{\sup}$ are positive continuous functions in Ω (possibly the constant $+\infty$), with $g_{\inf}$ being superharmonic in $\Omega\backslash\{\infty\}$, and $g_{\sup}$ subharmonic in $\Omega\backslash\{\infty\}$. These properties are illustrated in Example 2.9. Connections to the Green function of the resolvent set are given in Theorem 2.10. In particular, we are interested in Corollary 2.12 to characterize the so-called regular case where $g_{\sup} = g_{\inf}$ coincide with the Green function.

In Section 4 we investigate the question of local uniform convergence of *J*-fractions with complex bounded coefficients. Obviously, poles are obstacles for such a convergence, however, as observed by Gonchar [12], the absence of poles in some (particular) domain may already imply local uniform convergence. In Theorem 4.1 we show that any subdomain of Ω_0 has such properties. Some special cases are discussed in Corollary 4.2 and Corollary 4.3, for instance the case of operators with spectrum having capacity zero.

For the sharpness of the above convergence assertions, it is of interest to know whether the limit, namely the Weyl function of the difference operator A, has an analytic continuation in any domain strictly larger than Ω_0. This is known to be false for many special cases (like self-adjoint operators or periodic operators). In the final Section 5 we relate in Theorem 5.3 isolated points of the spectrum to poles or essential singularities of the Weyl function. Some implications for the analytic or meromorphic continuation of the Weyl function are discussed in Corollary 5.4 and Corollary 5.5. Finally, we show in Corollary 5.6 that the Baker–Gammel–Wills conjecture holds in the case of a countable spectrum, namely, there is local uniform convergence of a subsequence of Padé approximants in the maximal domain of analyticity of the Weyl function.

2. ASYMPTOTICS FOR FORMAL ORTHONORMAL POLYNOMIALS

For a proof of the convergence assertions presented in the second part of this paper we require several properties of the Padé denominators, i.e., of the formal orthonormal polynomials $(q_n)_{n \geqslant 0}$. In what follows, we denote by $v(f, F)$ for some closed $F \subset \overline{\mathbb{C}}$ the number of zeros (counting multiplicities) of some function f analytic on F. Furthermore, we require the leading coefficient of q_n, which according to (3), (4), is given by

$$k_n := \frac{1}{a_0 \cdot a_1 \cdot \cdots \cdot a_{n-1}}, \qquad \kappa_{\sup} := \limsup_{n \to \infty} |k_n|^{-1/n}, \qquad \kappa_{\inf} := \liminf_{n \to \infty} |k_n|^{-1/n}. \tag{11}$$

In addition, we denote the zero counting measure of q_n by μ_n, $n \geqslant 0$, with

$$|q_n(z)|^{1/n} = |k_n|^{1/n} \cdot \exp(-V[\mu_n](z)),$$

where $V[\mu]$ denotes the logarithmic potential of a positive Borel measure μ. The asymptotic properties of the Padé denominators will be stated in terms of the two functions

$$v_n(z) := \sqrt{|q_n(z)|^2 + |a_n \cdot q_{n+1}(z)|^2}, \qquad u_n(z) := \frac{q_n(z)}{a_n \cdot q_{n+1}(z)}, \qquad n \geqslant 0. \tag{12}$$

2.1. *Zero Distribution and Ratio Asymptotics*

Widom [25] observed for the case of real recurrence coefficients that the number of spurious poles (i.e., of zeros of q_n in Ω_0) is not arbitrary. As our first result, we show that this property remains valid for complex recurrence coefficients. This result is the key for establishing convergence in capacity.

PROPOSITION 2.1. *For every closed $F \subset \Omega_0$ there exists an $v(F)$ so that the number $v(q_n, F)$ of spurious poles in F is bounded by $v(F)$ for all $n \geqslant 0$. Moreover, the same property holds for the sequence of analytic functions $(z \cdot r_n \cdot q_n)_{n \geqslant 0}$. In addition, there exists a closed neighborhood U of ∞ with $v(z \cdot r_n \cdot q_n, U) = 0$ for all $n \geqslant 0$.*

Before giving a proof, let us motivate and state the second main result of this subsection. The sequence $(1/u_n)_{n \geqslant 0}$ of meromorphic functions is referred to as a tail sequence of the continued fraction [15, Sect. II.1.2, Eq. (1.2.7)], since by (11) we have $1/u_n = \tilde{q}_{n+1}/\tilde{q}_n$, with $\tilde{q}_n = q_n/k_n$ which is monic. In case that $(u_n)_{n\geqslant 0}$ converges in some set, one usually speaks of ratio asymptotic behavior of the corresponding sequence $(\tilde{q}_n)_{n\geqslant 0}$. In general we may not expect such a behavior for our setting; however, suitable subsequences of $(u_n)_{n\geqslant 0}$ will converge.

PROPOSITION 2.2. *The sequences* $(u_n)_{n\geqslant 0}$ *and* $(q_n/q_{n+1})_{n\geqslant 0}$ *of meromorphic functions are normal in* Ω *with respect to the chordal metric on the Riemann sphere. Furthermore, any limit function of* $(u_n)_{n\geqslant 0}$ *in the outer domain* Ω_0 *is different from the constants* 0 *and* ∞.

The proofs of the above assertions as well as other results presented in this work rely very much on a paper by Aptekarev *et al.* [2] on bounded (not necessarily self-adjoint) second order difference operators A (see also [7, 8] for recent complements and improvements). In [2, Theorem 1], a characterization of $\Omega(A)$ is given in terms of Padé polynomials p_n, q_n; moreover, the authors provide an explicit representation of the corresponding resolvent operator in terms of $(r_n)_{n\geqslant 0}$, $(q_n)_{n\geqslant 0}$, together with upper bounds for the elements of this operator. By slightly rephrasing [2, Eq. (12)] (see also [8, Theorem 2.1]) we get for $z\in\Omega(A)$ and for $j, k=0, 1, ...$

$$(e_k, R(z)\, e_j)=\begin{cases} r_j(z)\cdot q_k(z) & \text{if} \quad j\geqslant k, \\ r_k(z)\cdot q_j(z) & \text{if} \quad j\leqslant k. \end{cases} \tag{13}$$

Using results of Demko *et al.* [10] on inverses of band matrices, it is shown in [2, Theorem 1] that for every $z\in\Omega$ there exist positive constants $\beta(z)$ and $\delta(z)$ such that for all $0\leqslant j\leqslant k$

$$|r_k(z)\cdot q_j(z)|\leqslant \beta(z)\cdot\delta(z)^{k-j}, \qquad \delta(z)<1. \tag{14}$$

The continuity of β, δ as well as some further estimates are discussed in the Lemma below, we closely follow arguments from [8, Lemma 3.3].

LEMMA 2.3. *We may choose functions* β, δ *in* (14), *continuous in* Ω (*including infinity*), *so that* $\beta(z)\cdot|z|$ *is continuous at infinity. Furthermore, there exists a positive function* γ *continuous in* Ω (*including infinity*) *such that for all* $z\in\Omega$ *and for all* $n\geqslant 0$

$$1\leqslant v_n(z)\cdot\sqrt{|r_n(z)|^2+|a_n\cdot r_{n+1}(z)|^2}\leqslant\gamma(z), \tag{15}$$

$$1\leqslant |a_n|\cdot\sqrt{|q_n(z)|^2+|q_{n+1}(z)|^2}\cdot\sqrt{|r_n(z)|^2+|r_{n+1}(z)|^2}\leqslant\gamma(z). \tag{16}$$

Moreover,

$$\limsup_{n\to\infty}|r_n(z)|^{1/(n+1)}=e^{-g_{\inf}(z)}<1, \qquad z\in\Omega. \tag{17}$$

Proof. As shown in [8, Lemma 3.3], functions β_1, δ_1 satisfying (14) and continuous in $\Omega\setminus\{\infty\}$ result immediately from [10, Theorem 2.4]. In order to include the point ∞, let $D := \{z \in \overline{\mathbb{C}}: |z| \geqslant 2 \cdot \|A\|\}$. Obviously, the closure of D is a subset of Ω. By continuity, the two functions

$$\beta_2(z) := \frac{2 \cdot \|A\|}{|z|} \cdot \max_{z \in \partial D} \beta_1(z), \qquad \delta_2(z) := \max_{z \in \partial D} \delta_1(z) < 1,$$

verify (14) for $z \in \partial D$. From (5), we see that $z \cdot r_k(z) \cdot q_j(z)$ is analytic in a neighborhood of the closure of D including infinity for all $0 \leqslant j \leqslant k$. Thus, by the maximum principle for analytic functions, relation (14) also holds for $z \in D$ for the functions β_2, δ_2. Define

$$\log \beta(z) := \begin{cases} \log \beta_1(z) & \text{if} \quad z \in \Omega, \quad |z| \leqslant 2 \cdot \|A\|, \\ \log \beta_2(z) & \text{if} \quad |z| \geqslant 3 \cdot \|A\|, \\ \left(3 - \dfrac{|z|}{\|A\|}\right) \cdot \log \beta_1(z) + \left(\dfrac{|z|}{\|A\|} - 2\right) \cdot \log \beta_2(z) & \\ & \text{if} \quad 2 \cdot \|A\| \leqslant |z| \leqslant 3 \cdot \|A\|, \end{cases}$$

and $\delta(z)$ by a similar combination of $\delta_1(z)$ and $\delta_2(z)$. Then β, δ satisfy (14) and have the desired regularity properties.

For a proof of (15) and (16), one first verifies by recurrence using (3), (4), and (5), that, for $n \geqslant 0$,

$$a_n \cdot (q_n \cdot p_{n+1} - q_{n+1} \cdot p_n) = 1, \tag{18}$$

$$a_n \cdot (r_n \cdot q_{n+1} - r_{n+1} \cdot q_n) = 1. \tag{19}$$

Thus the lower bounds in (15) and (16) follow by applying the Cauchy–Schwarz inequality on (19). For the upper bounds, we first replace the term $|a_n| \cdot |q_{n+1}(z)|$ by its upper bound $(|z| + |b_n|) \cdot |q_n(z)| + |a_{n-1}| \cdot |q_{n-1}(z)|$. In the case (16), we obtain

$$\begin{aligned} &|a_n| \cdot \sqrt{|q_n(z)|^2 + |q_{n+1}(z)|^2} \cdot \sqrt{|r_n(z)|^2 + |r_{n+1}(z)|^2} \\ &\quad \leqslant (|z| + |a_n| + |b_n|) \cdot (|q_n(z)\, r_n(z)| + |q_n(z)\, r_{n+1}(z)|) \\ &\qquad + |a_{n-1}| \cdot (|q_{n-1} q(z)\, r_n(z)| + |q_{n-1}(z)\, r_{n+1}(z)|), \end{aligned}$$

and, by (14), we may bound the right hand side, e.g., by $(2 \cdot |z| + 6 \cdot \|A\|) \cdot \beta(z)$, which is continuous in Ω (including ∞). A proof for inequality (15) is similar, we omit the details.

It remains to establish (17). By taking $j=0$ in (14) we get

$$1 > \delta(z) \geqslant \limsup_{n\to\infty} |r_n(z)|^{1/(n+1)}$$
$$= \limsup_{n\to\infty} [\,|r_n(z)|^2 + |a_n \cdot r_{n+1}(z)|^2\,]^{1/(2n+2)},$$

the equality being a consequence of the fact that $(a_n)_{n\geqslant 0}$ is bounded. On the other hand, by (15),

$$\limsup_{n\to\infty} [\,|r_n(z)|^2 + |a_n \cdot r_{n+1}(z)|^2\,]^{1/(2n+2)}$$
$$= \limsup_{n\to\infty} v_n(z)^{-1/(n+1)}, \qquad z \in \Omega,$$

and the assertion follows from (9). ∎

A sequence of functions analytic in some domain $D \subset \overline{\mathbb{C}}$ is called normal in D if, given a subsequence, we may extract a subsequence converging locally uniformly in D with respect to the usual Euclidean metric, and thus the limit function is analytic by theorem of Weierstrass. Recall that, by a theorem of Montel[1] a family of functions analytic in D is normal in D if and only if it is uniformly bounded on any closed subset of D.

In this paper (see, e.g., Proposition 2.2), we also deal with sequences of functions which are meromorphic in some domain D. Such a sequence is called normal in D if, given a subsequence, we may extract a subsequence converging locally uniformly in D with respect to the chordal metric $\chi(\cdot)$ on the Riemann sphere [17, Definition 3.1.1]. Here, any limit function is either meromorphic in D or identically ∞ [17, Corollary 3.1.4]. If confusion is possible, we will explicitly mention the corresponding metric. Notice however that a normal family of analytic functions also is normal with respect to the chordal metric. A reciprocal statement is discussed in Lemma 2.4(d) below.

In what follows we will use several classical properties of normal families, for instance some reformulation of a theorem of Hurwitz and a link between normal families of analytic and of meromorphic functions. For the sake of completeness, these properties are stated (and proved) in

LEMMA 2.4. *Let $D \subset \overline{\mathbb{C}}$ be some domain, and suppose that the sequence $(f_n)_{n\geqslant 0}$ of functions analytic in D is normal (with respect to the usual*

[1] See e.g., [17, Sect. 2.2 and Theorem 2.2.2]. Some authors prefer to measure the distance of functions in the chordal metric, in which case a subsequence may also converge to the constant ∞. However, this case must be excluded in order to obtain equivalence in the Montel Theorem.

Euclidean metric). *Furthermore, suppose that all limit functions are different from the constant* 0. *Then there holds*

(a) (*See* [17, *Theorem* 2.5.1].) *For any closed* $F \subset D$, *the number of zeros of* f_n, *in* F *is uniformly bounded in* n: *there exists a* $\nu(F)$ *such that* $\nu(f_n, F) \leqslant \nu(F)$ *for all* $n \geqslant 0$.

(b) *If* $f_n(\zeta) = 1$, $n \geqslant 0$, *for some* $\zeta \in D$, *then there exists a closed neighborhood* $F \subset D$ *of* ζ *with* $\nu(f_n, F) = 0$ *for all* $n \geqslant 0$.

(c) *Let* $F \subset D$ *be some closed set. If all functions* f_n *are different from zero in some open neighborhood of* F *then there exists a constant* $C > 0$ *such that* $C \leqslant |f_n(z)|$ *for all* $n \geqslant 0$ *and for all* $z \in F$. *In the general case, provided that* $\infty \notin F$, *there exist constants* C, ν *and monic polynomials* $\hat{f}_n$, $n \geqslant 0$, *of degree at most* ν *such that*

$$C \cdot |\hat{f}_n(z)| \leqslant |f_n(z)|, \qquad n \geqslant 0, \quad z \in F.$$

(d) (*Compare with* [17, *Corollary* 3.1.7].) *Let* h_n, $n \geqslant 0$, *be meromorphic in* D, *and suppose that* $(h_n)_{n \geqslant 0}$ *is normal in* D *with respect to the chordal metric, with any limit function being different from the constant* ∞. *If there is a domain* $D' \subset D$ *with* h_n *being analytic in* D' *for* $n \geqslant 0$, *then* $(h_n)_{n \geqslant 0}$ *is normal in* D' *with respect to the Euclidean metric.*

Proof. (a) If the first part of the assertion fails to hold, then by taking a subsequence we may suppose that $\nu(f_n, F) \geqslant n$ for all $n \geqslant 0$, and that $(f_n)_{n \geqslant 0}$ converges locally uniformly in D to some f. Then f is analytic in D and, by assumption, f is not identically zero. Thus the zeros of f do not accumulate in D. By possibly slightly enlarging F we may insure that f is different from zero on ∂F, and thus $\varepsilon := \min_{z \in \partial F} |f(z)| > 0$. In view of the uniform convergence of $(f_n)_{n \geqslant 0}$ on F, we have for sufficiently large n and for every $z \in \partial F$

$$|f(z) - f_n(z)| < \varepsilon \leqslant |f(z)|.$$

From the Rouché Theorem it follows that $\nu(f_n, F) = \nu(f, F) < \infty$ for sufficiently large n, contradiction to the choice of the sequence $(f_n)_{n \geqslant 0}$.

(b) By the Arzela–Ascoli Theorem, the normality of the sequence $(f_n)_{n \geqslant 0}$ implies in particular that f_n, $n \geqslant 0$, are equicontinuous. Thus there is a $\delta > 0$ such that for all $n \geqslant 0$ and for all z with $\chi(z, \zeta) < \delta$ there holds $|f_n(z) - f_n(\zeta)| \leqslant 1/2$, and therefore $|f_n(z)| \geqslant 1/2$.

(c) For a proof of the first part, let D_1 be some domain containing F with closure $F_1 \subset D$ such that $f_n(z) \neq 0$ for all $n \geqslant 0$ and for all $z \in D_1$. If $(f_n)_{n \geqslant 0}$ is not bounded away from zero uniformly on F then there exists a limit function f having a zero in F. It follows from the Theorem of Hurwitz that f vanishes identically in D_1, in contradiction with the assumption of Lemma 2.4. This shows the first assertion of part (c).

In order to show the second one, let $D_1, D_2 \subset D$ be some domains withclosures F_1, and F_2, respectively, verifying $F \subset D_1$, $F_1 \subset D_2$, $F_2 \subset D$, and $\infty \notin F_2$. Furthermore, $v = v(F_1) < \infty$ as in part (a). For $n \geqslant 0$, we denote by $z_1, ..., z_{v(f_n, F_1)}$ the (finite) zeros of f_n, in F_1, (counting multiplicities), and define $\hat{f}_n(z) = (z - z_1) \cdot \cdots \cdot (z - z_{v(f_n, F_1)})$, $g_n := f_n / \hat{f}_n$. By the Theorem of Montel, the sequence $(f_n)_{n \geqslant 0}$ is uniformly bounded on F_2 by some constant C'. Moreover, by construction, g_n is analytic in D, and $(g_n)_{n \geqslant 0}$ is uniformly bounded on the boundary of F_2 by $C' \cdot \max\{1, dist(\partial F_1, \partial F_2)^{-v}\} < \infty$. From the maximum principle for analytic functions, we may conclude that $(g_n)_{n \geqslant 0}$ is normal in D_2. In addition, each function g_n is different from zero in the neighborhood D_1 of F by construction, and each limit function of $(g_n)_{n \geqslant 0}$ is different from the constant 0 by assumption on $(f_n)_{n \geqslant 0}$. Hence the assertion is a consequence of the first part of (c).

(d) Suppose that assertion (d) is not true. From the Montel theorem, we may conclude that there exists a closed set $F \subset D'$ with $(h_n)_{n \geqslant 0}$ not being uniformly bounded on F. By possibly taking subsequences, we find a sequence $(z_n)_{n \geqslant 0} \subset F$ converging to some $\zeta \in D'$ and $(|h_n(z_n)|)_{n \geqslant 0}$ tending to infinity. By the equicontinuity of $(h_n)_{n \geqslant 0}$ on F with respect to the chordal metric, there exist a $\delta > 0$ and an $N > 0$ such that $\chi(h_n(z), \infty) < 1/2$ for all $n \geqslant N$ and for all $z \in U := \{t \in D' : \chi(t, \zeta) < \delta\}$. Consequently, for $n \geqslant N$, the function $1/h_n$ is analytic in U and bounded in modulus by 1, and thus $(1/h_n)_{n \geqslant N}$ is normal in U with respect to the Euclidean metric. Also, by assumption, $1/h_n$, $n \geqslant 0$, is different from zero in U. Denoting by h a limit function, we have by construction $h(\zeta) = 0$, and thus by the Theorem of Hurwitz $h = 0$ in U. Since $\chi(1/h_n(z), 0) = \chi(h_n(z), \infty)$, by taking again subsequences we find a limit function of $(h_n)_{n \geqslant 0}$ in D with respect to the chordal metric which is equal to the constant ∞ in U, and thus in D. This contradicts the hypothesis of Lemma 2.4(d). ∎

Proof of Proposition 2.1. We consider the functions $g_n(z) := r_n(z) \cdot q_n(z)$, and $f_n(z) := z \cdot g_n(z)$, $n \geqslant 0$. From (13) we see that

$$g_n(z) = (R(z)\, e_n, e_n) = \sum_{j=0}^{\infty} \frac{(A^j e_n, e_n)}{z^{j+1}} = \frac{1}{z} + \mathcal{O}\left(\frac{1}{z^2}\right)_{z \to \infty}.$$

In particular, $(g_n)_{n \geqslant 0}$ and $(f_n)_{n \geqslant 0}$ are sequences of functions analytic in Ω. According to (14) and Lemma 2.3 they are locally uniformly bounded in Ω and thus normal in Ω by the Montel Theorem. In addition, $f_n(\infty) = 1$ for all $n \geqslant 0$, showing that none of the limit functions of $(f_n)_{n \geqslant 0}$ is identically zero in the domain $D := \Omega_0$. Thus, the assertion follows from Lemma 2.4(a), (b). ∎

Proof of Proposition 2.2. According to the Marty Theorem, for the normality of $(u_n)_{n\geqslant 0}$ in some domain $D\subset\mathbb{C}$ it is sufficient to show that the spherical derivative

$$\rho(u_n):=\frac{|u_n'|}{1+|u_n|^2}$$

is bounded uniformly with respect to n on compact subsets of D. We have

$$\begin{aligned}\rho(u_n)(z)&=\frac{|(q_n'(z)\,q_{n+1}(z)-q_n(z)\,q_{n+1}'(z))/a_n\cdot q_{n+1}(z)^2|}{1+|q_n(z)/a_n\cdot q_{n+1}(z)|^2}\\&=\frac{|a_n|\cdot|q_n'(z)\,a_{n+1}(z)-q_n(z)\,q_{n+1}'(z)|}{v_n(z)^2}.\end{aligned}$$

One easily establishes by recurrence using (3), (4), the Christoffel–Darboux formula

$$a_n\cdot\frac{q_n(x)\,q_{n+1}(z)-q_n(z)\,q_{n+1}(x)}{z-x}=\sum_{j=0}^{n}q_j(x)\cdot q_j(z),$$

which in the limit $x\to z$ takes the form

$$a_n\cdot[q_n(z)\,q_{n+1}'(z)-q_n'(z)\,q_{n+1}(z)]=\sum_{j=0}^{n}q_j(z)^2.$$

Consequently, using (15), we get

$$\begin{aligned}\rho(u_n)(z)&\leqslant\frac{1}{v_n(z)^2}\cdot\sum_{j=0}^{n}|q_j(z)|^2\\&\leqslant[\,|r_n(z)|^2+|a_n\cdot r_{n+1}(z)|^2]\cdot\sum_{j=0}^{n}|q_j(z)|^2.\end{aligned}$$

Here, the right hand side may be estimated using (14) by $(1+\|A\|^2)\cdot\beta(z)^2/(1-\delta(z)^2)$, which by Lemma 2.3 is continuous in Ω. Consequently, $(\rho(u_n))_{n\geqslant 0}$ is bounded locally uniformly in $\Omega\backslash\{\infty\}$. In order to include a neighborhood of infinity, we consider the rational functions $\tilde{u}_n(z):=u_n(1/z)$ and observe that

$$\rho(\tilde{u}_n)(1/z)=|z|^2\cdot\rho(u_n)(z),$$

where again the right hand side is bounded uniformly with respect to n in some neighborhood of infinity. Therefore, $(u_n)_{n\geqslant 0}$ is normal in Ω.

The normality of $(q_n/q_{n+1})_{n\geqslant 0}$ in Ω can be established using similar arguments and (16), since

$$\rho\left(\frac{q_n}{q_{n+1}}\right)(z)=\frac{1}{|a_n|\cdot(|q_n(z)|^2+|q_{n+1}(z)|^2)}\cdot\left|\sum_{j=0}^{n} q_j(z)^2\right|.$$

It remains to show the characterization of the limit functions of $(u_n)_{n\geqslant 0}$ in Ω_0. We know from (3), (4), and (12) that

$$u_n(z)=\frac{1}{z}+\mathcal{O}(z^{-2})_{z\to\infty},\qquad n\geqslant 0,$$

i.e. $u_n(\infty)=0$ and $u_n'(\infty)=1$ for all $n\geqslant 0$. This implies that any limit function u of $(u_n)_{n\geqslant 0}$ is analytic in some neighborhood of infinity, with $u'(\infty)=1$ by the Weierstrass Theorem. In particular, u is different from a constant in Ω_0. ∎

2.2. *nth-Root Asymptotic Behavior*

We know from (9), (10), and (12) that $g_{\inf}$ is the (pointwise) lower limit of the sequence $(\log v_n^{1/n})_{n\geqslant 0}$. Moreover, since A is bounded, $g_{\sup}$ is the (pointwise) upper limit of the sequence $(\log v_n^{1/n})_{n\geqslant 0}$. For self-adjoint operators A, properties of these two functions may be derived by specifying results given in the monograph [20] on general orthogonal polynomials. In general case, Aptekarev *et al.* showed [2, Corollary 3] that $g_{\sup}$ is positive in Ω, and some properties of a function closely related to $g_{\inf}$ have been investigated by Kaliaguine and the present author [8, Theorem 3.6].

THEOREM 2.5. *Let $\omega\notin\Omega$.*

(a) *The sequence $(v_n^{-1/n})_{n\geqslant 0}$ of continuous functions is normal in Ω (with respect to Euclidean metric). We denote by $\mathcal{G}$ the set of limit functions of $(\log v_n^{1/n})_{n\geqslant 0}$.*

(b) *Let $g\in\mathcal{G}$, say,*

$$g(z)=\lim_{n\to\infty,\, n\in\Omega}\log v_n(z)^{1/n},\qquad z\in\Omega,$$

for some infinite set $\Lambda\subset\mathbb{N}$. Then the limit

$$\kappa_\Lambda:=\lim_{n\to\infty,\, n\in\Lambda}|k_n|^{-1/n}$$

exists, and g is equal to the constant $+\infty$ iff $\kappa_\Lambda=0$. Otherwise, $g(z)-\log|z-\omega|$ is harmonic in Ω,

$$g(z)\leqslant\log\frac{1}{\kappa_\Lambda}+\log(|z|+\|A\|),\qquad z\in\Omega,$$

and in the outer component of the resolvent set, we have the representation

$$g(z) = \log \frac{1}{\kappa_\Lambda} - V[\mu](z), \qquad z \in \Omega_0,$$

where μ denotes any weak limit of $(\mu_n)_{n\in\Lambda} \cup (\mu_{n+1})_{n\in\Lambda}$.

(c) *The limit relations*

$$\limsup_{n\to\infty} v_n(z)^{-1/n} = \exp(-g_{\inf}(z)), \qquad \liminf_{n\to\infty} v_n(z)^{-1/n} = \exp(-g_{\sup}(z))$$

are valid locally uniformly in Ω. Furthermore, $g_{\inf}$, $g_{\sup}$ are positive in Ω (possibly the constant $+\infty$), with $g_{\inf}(z) - \log|z-\omega|$ being superharmonic and continuous in Ω, and $g_{\sup}(z) - \log|z-\omega|$ subharmonic and continuous in Ω.

Notice that, as a consequence of (15), the sequences $(v_n^{-1/n})_{n\geqslant 0}$ and $([|z_n(z)|^2 + |a_n \cdot r_{n+1}(z)|^2]^{1/(2n)})_{n\geqslant 0}$ have the same asymptotic behavior; in particular, it is possible to restate analogues of Theorem 2.5 for residuals.

For the proof of Theorem 2.5 we require some preliminary considerations. In the next three lemmas, we denote by $B(\zeta, \delta) = \{z \in \overline{\mathbb{C}}: \chi(z, \zeta) \leqslant \delta\}$ the closed chordal disk with center $\zeta \in \overline{\mathbb{C}}$ and radius $\delta > 0$.

LEMMA 2.6. *For any closed $F \subset \Omega$ there exists a $\delta > 0$ with the following properties: for any $\zeta \in F$ we have $B(\zeta, \delta) \subset \Omega$, and there exists a sequence $(\varepsilon_n)_{n\geqslant 0} \subset \{0, 1\}$ (in general, depending on ζ) such that*

$$-\frac{\log(4)}{n} \leqslant \log(|k_n/v_n|^{1/n}) - \frac{n+\varepsilon_n}{n} \cdot V[\mu_{n+\varepsilon_n}](z) \leqslant 0, \quad n \geqslant 0, \quad z \in B(\zeta, \delta),$$

and $v(q_{n+\varepsilon_n}, B(\zeta, \delta)) = 0$ for all $n \geqslant 0$. In addition, if $\infty \in B(\zeta, \delta)$, then $\varepsilon_n = 1$ for all n.

Proof. First, there holds $B(\zeta, \delta) \subset \Omega$ for all $\zeta \in F$ if δ is smaller than the spherical distance of the boundaries of Ω and F. From Proposition 2.2, we know that $(u_n)_{n\geqslant 0}$ is normal in Ω and thus equicontinuous on closed subsets of Ω. Hence, by possibly choosing a smaller δ, we may insure that $\chi(u_n(z), u_n(\zeta)) < 1/4$ for all $n \geqslant 0$, for all $\zeta \in F$, and for all $z \in B(\zeta, \delta)$. Notice that

$$\chi(0, u_n(z)) = \frac{|k_n/v_n(z)|}{|k_n/q_n(z)|}, \qquad \chi(\infty, u_n(z)) = \frac{|k_n/v_n(z)|}{|k_{n+1}/q_{n+1}(z)|}.$$

Given a fixed $\zeta \in F$, we may choose $\varepsilon_n = 0$ if $\chi(0, u_n(\zeta)) \geqslant 1/2$ and $\varepsilon_n = 1$ otherwise (i.e., if $\chi(\infty, u_n(\zeta)) > 1/2$). In particular, if $\infty \in B(\zeta, \delta)$, then $\varepsilon_n = 1$ for all n since $u_n(\infty) = 0$. This yields

$$\frac{1}{4} \leqslant \frac{|k_n/v_n(z)|}{|k_{n+\varepsilon_n}/q_{n+\varepsilon_n}(z)|} \leqslant 1, \qquad z \in B(\zeta, \delta),$$

and the estimate in the assertion follows by taking logarithms and by dividing by n. Finally, by (18) we have $v_n(z) \neq 0$ for all $z \in \mathbb{C}$, and thus $q_{n+\varepsilon_n}(z) \neq 0$ for all $n \geqslant 0$ and for all $z \in B(\zeta, \delta)$. ∎

In addition we have a statement similar to [12, Proposition 4].

LEMMA 2.7. *Let $\Lambda \subset \mathbb{N}$ be infinite, $\varepsilon_n \in \{0, 1\}$, and suppose that the sequence of measure $(\mu_{n+\varepsilon_n})_{n \in \Lambda}$ and $(\mu_{n+1-\varepsilon_n})_{n \in \Lambda}$ converge weakly to $\mu^{(0)}$, and to $\mu^{(1)}$, respectively. Then the potentials of both limit measures coincide in Ω_0.*

Proof. First notice that, by Proposition 2.1, both measures are supported on the complement of Ω_0, and thus the potentials are continuous in Ω_0. We may find a set $\Lambda_1 \subset \Lambda$ such that $(u_n)_{n \in \Lambda_1}$ converges to some meromorphic function u locally uniformly in Ω_0 with respect to the chordal metric. According to the fact that u is different from the constants 0, ∞, the sequence $(|u_n|^{1/n})_{n \in \Lambda_1}$ converges pointwise to 1 quasi everywhere[2] in Ω_0. Since

$$\log |u_n|^{1/n} = \frac{n+1}{n} V[\mu_{n+1}] - V[\mu_n], \qquad n \geqslant 0,$$

the assertion follows. ∎

Our proof of Theorem 2.5 is essentially based on Lemma 2.6 and the following

LEMMA 2.8. *Let $\omega \notin \Omega$ be some fixed complex number. The sequence $(\log v_n(z)^{1/(n+1)} - \log |z - \omega|)_{n \geqslant 0}$ of continuous functions is equicontinuous in Ω with respect to Euclidean metric, and locally uniformly bounded below in Ω.*

Proof. Let

$$w_n(z) := \log(|z-\omega| \cdot |k_n/v_n(z)|^{1/(n+1)}), \qquad n \geqslant 0.$$

In the first part of the proof, we want to show that the sequence $(w_n)_{n \geqslant 0}$ of continuous functions is normal in Ω, with any limit function w being

[2] A property is said to hold quasi everywhere in some set D if it is true in $D \backslash K$, where K has capacity zero.

harmonic in Ω and vanishing at infinity. Consider an exhaustion of Ω, namely, a sequence of open sets $D_j \subset \Omega$, $j \geqslant 0$, with its closure F_j being a subset of D_{j+1}, and $\bigcup_j D_j = \Omega$. Furthermore, we denote by $\delta_j > 0$ the number associated with F_j as described in Lemma 2.6. Notice that every F_j may be covered by a finite number of open chordal disks of radius $\delta_j/2$ centered in F_j. Therefore it is sufficient to show that, given a $\zeta \in F_j$, we may extract from every sequence $(w_n)_{n \in \Lambda}$ a subsequence converging uniformly in $B(\zeta, \delta_j/2)$ to some function w which is harmonic in the interior of $B(\zeta, \delta_j/2)$.

Given a $\zeta \in F_j$, let $(\varepsilon_n)_{n \geqslant 0}$ be as in Lemma 2.6, and write shorter $B := B(\zeta, \delta_j)$, $B' := B(\zeta, \delta_j/2)$. From the last of Proposition 2.1, we know that there exists some compact set K containing the supports of the measure $\mu_{n+\varepsilon_n}$ for all $n \geqslant 0$. Helly's theorem asserts that any sequence $(\mu_{n+\varepsilon_n})_{n \in \lambda}$ contains a subsequence $(\mu_{n+\varepsilon_n})_{n \in \Lambda'}$ that converges weak* to some probability measure μ. According to Lemma 2.6, $\mu_{n+\varepsilon_n}$ is supported in $K \backslash B$. Therefore, μ is supported in the closure of $K \backslash B$, a subset of the complement of B'. In particular, the sequence of potentials $(V[\mu_{n+\varepsilon_n}](z) + \log |z-\omega|)_{n \in \Lambda'}$ converges uniformly to $w(z) := V[\mu](z) + \log |z - \omega|$ on B'. Thus, with the aid of Lemma 2.6 we obtain the uniform convergence of $(w_n)_{n \in \Lambda'}$ to w on B'. Finally, notice that w is harmonic in the interior of B', and $w(\infty) = 0$. This proves the assertion made in the beginning of the proof.

From the Arzela–Ascoli Theorem, we may conclude that $(w_n)_{n \geqslant 0}$ is equicontinuous and locally uniformly bounded in Ω. Thus the assertion of the lemma follows immediately since

$$\log v_n(z)^{1/n+1} = \log |z-\omega| - w_n(z) + \frac{n}{n+1} \cdot \log \frac{1}{|k_n|^{-1/n}}, \tag{20}$$

and $|k_n|^{-1/n} \leqslant \|A\|$ for all n. ∎

It is known that all the zeros of q_n have a modulus less than or equal to $\|A\|$ (see, e.g., [8, Theorem 3.10]), and thus

$$V[\mu_n](z) \geqslant \log \frac{1}{|z| + \|A\|}, \qquad z \in \overline{\mathbb{C}}. \tag{21}$$

This estimate will enable us to derive explicit upper bounds for $(\log v_n(z)^{1/(n+1)} - \log |z-\omega|)_{n \geqslant 0}$.

Proof of Theorem 2.5. Since $v_n(z)^{-1/(n+1)} = |z-\omega|^{-1} \cdot \exp(-[\log v_n(z)^{1/(n+1)} - \log |z-\omega|])$, we may conclude from Lemma 2.8, that the sequence $(v_n^{-1/n})_{n \geqslant 0}$ is equicontinuous in Ω, and locally uniformly bounded above. Also, by construction, this sequence is trivially bounded below by zero in Ω, and assertion (a) is a consequence of the Arzela–Ascoli Theorem.

For a proof of part (b), let $\Lambda' \subset \Lambda$ be such that $(|k_n|^{-1/n})_{n\in\Lambda'}$ tends to some $\kappa \in [0, \|A\|]$, and $(w_n)_{n\in\Lambda'}$ converges locally uniformly in Ω to some w. We know from Lemma 2.8 that w is harmonic in Ω, vanishes at ∞, and $w(z) \geqslant \log[(|z-\omega|)/(|z|+\|A\|)]$ for $z\in\Omega$ by (21). According to the fact that $(w_n)_{n\in\Lambda'}$ is bounded uniformly on closed subsets of Ω, we see from (20) that $(\log v_n^{1/n})_{n\in\Lambda'}$ tends to the constant $+\infty$ locally uniformly in Ω if and only if $\kappa=0$. Otherwise, $(\log v_n^{1/(n+1)})_{n\in\Lambda'}$ converges locally uniformly in Ω to $g(z) := \log|z-\omega| - w(z) + \log(1/\kappa)$. In order to deduce the representation of g in Ω_0, we may in addition assume that both sequences of zero counting measures $(\mu_n)_{n\in\Lambda'}$ and $(\mu_{n+1})_{n\in\Lambda'}$ converge weakly. Notice that the potentials of the limit measures coincide in Ω_0 by Lemma 2.7. Thus Lemma 2.8 yields the desired representation $w(z) = \log|z-\omega| + \mu V[\mu](z)$ for, e.g., the limit μ of $(\mu_n)_{n\in\Lambda'}$. It remains to show that all these findings do not depend on the choice of the subsequence Λ' of Λ. In fact, if $\Lambda^* \subset \Lambda$ is chosen so that $(\log v_n^{1/(n+1)})_{n\in\Lambda^*}$ converges locally uniformly in Ω to $g^*(z) := \log|z-\omega| - w^*(z) + \log(1/\kappa^*)$, then $g(z) = g^*(z)$ by assumption on Λ, and from $w(\infty) = w^*(\infty) = 0$ we may conclude that $\kappa = \kappa^*$, and $w = w^*$ in Ω.

In order to show (c), notice first that

$$\log|q_n(z)|^{1/n} \leqslant \log|v_n(z)|^{1/n},$$

$$\log|v_n(z)|^{1/(n+1)} \leqslant 2^{1/(2n+2)} \cdot \max\left\{\log|q_n(z)|^{1/(n+1)}, \frac{1}{n+1}\cdot\log\|A\| \right.$$

$$\left. + \log|q_{n+1}(z)|^{1/(n+1)}\right\}.$$

Therefore, by (9), (10), the claimed limit relations hold pointwise for $z\in\Omega$, with $0 < g_{\inf}(z) \leqslant g_{\sup}(z)$ for $z\in\Omega$ according to (17). From the equicontinuity property established in Lemma 2.8, we may conclude that these limit relations (and thus also (9), (10)) hold locally uniformly in Ω, with $g_{\inf}(z) - \log|z-\omega|$ and $g_{\sup}(z) - \log|z-\omega|$ being continuous in Ω, $\omega\notin\Omega$. Finally, again by equicontinuity and by part (a) we have for $z\in\Omega$

$$g_{\inf}(z) - \log|z-\omega| = \min\{g(z) - \log|z-\omega| : g\in\mathcal{G}\}, \tag{22}$$

$$g_{\sup}(z) - \log|z-\omega| = \max\{g(z) - \log|z-\omega| : g\in\mathcal{G}\}, \tag{23}$$

where we recall from part (b) that $g - \log|z-\omega|$ is harmonic in Ω for all $g\in\mathcal{G}$. In particular, $g_{\inf}(z) - \log|z-\omega|$ is superharmonic in Ω, and $g_{\sup}(z) - \log|z-\omega|$ is subharmonic in Ω, as claimed in Theorem 2.5(c). ∎

Already from the case of self-adjoint operators A, we know that Theorem 2.5 may not be essentially improved. If the sequence of zero counting measures $(\mu_n)_{n \geqslant 0}$ converges, the limit μ (which by Proposition 2.1 is supported on $\mathbb{C} \backslash \Omega_0$) usually is referred to as *density-of-states measure* (see, e.g., [11, Definition 1.1 and following Remarks]). Let A, and $\tilde{A}$, result from the infinite tridiagonal matrix (7) with entries a_j, b_j, and $\tilde{a}_j$, $\tilde{b}_j$, respectively. If both A and $\tilde{A}$ are bounded self-adjoint operators, $\tilde{A}$ has a density-of-states measure μ, and

$$\lim_{n \to \infty} \frac{1}{n} \cdot \sum_{j=0}^{n-1} (|a_j - \tilde{a}_j| + |b_j - \tilde{b}_j|) = 0, \tag{24}$$

then it follows from the general result [11, Theorem 2.4] of Geronimo *et al.* that also A has the density-of-states measure μ. This enables us to discuss several special cases of Theorem 2.5.

Example 2.9. In this example we take $\tilde{a}_n = 1/2$, $\tilde{b}_n = 0$, $n \geqslant 0$. Here, $\tilde{q}_n$ coincides up to a scaling with the Chebyshev polynomial U_n of the second kind. In particular, the density-of-states measure of $\tilde{A}$ is given by the equilibrium measure of $[-1, 1] = \sigma(\tilde{A})$ (see, e.g., [16, Sect. II.9.2]); and thus $\tilde{g}_{\sup} = \tilde{g}_{\inf}$ coincides with the Green function of $\Omega_0(\tilde{A})$.

(a) As in [11, Example 4.4], let $a_n = \tilde{a}_n = 1/2$, $n \geqslant 0$, $b_n = 1$ if $10^k \leqslant n < 10^k + k$ for some integer k, and $b_n = 0$ otherwise. Here, not only the density-of-states measure but also the quantities $\kappa_{\inf} = \kappa_{\sup} = 1/2$ remain invariant. On the other hand it is shown in [11, Example 4.4] that $\sigma(A) = [-1, +2]$, i.e., we have found an operator A with $g_{\inf} = g_{\sup}$ being a Green function, but not that of Ω_0.

(b) The equality $\kappa_{\inf} = \kappa_{\sup}$ does not necessarily imply that $g_{\inf} = g_{\sup}$ in Ω_0. Otherwise, it would follow from Theorem 2.5(b) that there necessarily exists a density-of-states measure, in contradiction with [11, Example 4.1] where the following example was studied: $a_n = \tilde{a}_n = 1/2$, $n \geqslant 0$, $b_n = (-1)^k$ if $k \leqslant \log \log n < k+1$ for some integer k, and $b_n = 1$ otherwise.

(c) The existence of a density-of-states measure does not imply that $\kappa_{\sup} = \kappa_{\inf}$ (and thus we may have $g_{\sup} - g_{\inf} \neq 0$ in Ω), as it becomes clear from the following example: with some fixed $\alpha \in (0, 1)$, choose $b_n = \tilde{b}_n = 0$, $n \geqslant 0$, $a_n = \tilde{a}_n \cdot \alpha^n$ if $n = 2^k$ for some integer k, and $a_n = \tilde{a}_n = 1/2$ otherwise. First, from [11, Theorem 2.4], we may conclude that A has the same density-of-states measure as $\tilde{A}$. Furthermore, $1/k_n = 2^{-n} \cdot \alpha^{2^{k+1}-1}$ for $2^k < n \leqslant 2^{k+1}$, and consequently $(|k_n|^{-1/n})_{n \geqslant 0}$ has accumulation points dense in $[\alpha^2/2, \alpha/2]$, with $\kappa_{\inf} = \alpha^2/2$, $\kappa_{\sup} = \alpha/2$. Thus, combining Theorem 2.5(b), (9), and (10), we get $g_{\sup}(z) - g_{\inf}(z) = -\log(\alpha)$ for $z \in \Omega$.

2.3. *Regular Asymptotic Behavior*

Following [20], it is of interest to relate the functions $g_{\inf}$, $g_{\sup}$ to the Green function of the outer component of the resolvent set. Such a relation was given in Example 2.9 for some particular cases. The exact connection as well as further properties are stated in

THEOREM 2.10. (a) *The function $g_{\inf}$ (and $g_{\sup}$, respectively) is harmonic in $\Omega_0 \backslash \{\infty\}$ iff it coincides in Ω_0 with an element of $\mathscr{G}$. A similar statement holds for any other connected component of Ω.*

(b) *The following statements are equivalent:*

(i) *There exists an element $g \in \mathscr{G}$ with limit boundary values 0 quasi everywhere on $\partial\Omega_0$, i.e.,*

$$\lim_{z \to \zeta,\, z \in \Omega_0} g(z) = 0 \qquad \text{for quasi every } \zeta \in \partial\Omega_0.$$

(ii) *$g_{\inf}$ has limit boundary values 0 quasi everywhere on $\partial\Omega_0$.*

(iii) *$g_{\inf} = g_{\Omega_0}$ in Ω_0.*

In any of these cases, the resolvent set Ω is connected, i.e., $\Omega_0 = \Omega$.

(c) *The following alternative holds for $g_{\inf}$:*

(i) *$\kappa_{\sup} = 0$. Here, $g_{\inf}(z) = +\infty$ for all $z \in \Omega$.*

(ii) *$\operatorname{cap}(\partial\Omega_0) = \kappa_{\sup} > 0$. Here, $g_{\inf}(z) = g_{\Omega_0}(z) > 0$ for all $z \in \Omega = \Omega_0$.*

(iii) *$\operatorname{cap}(\partial\Omega_0) > \kappa_{\sup} > 0$. Here, $0 < g_{\inf}(z) < \infty$ for all $z \in \Omega \backslash \{\infty\}$, and $g_{\inf}(z) > g_{\Omega_0}(z)$ for all $z \in \Omega_0 \backslash \{\infty\}$.*

(d) *The following alternative holds for $g_{\sup}$:*

(i) *$\kappa_{\inf} = 0$. Here, $g_{\sup}(z) = +\infty$ for all $z \in \Omega$.*

(ii) *$\operatorname{cap}(\partial\Omega_0) = \kappa_{\inf} > 0$. Here, $g_{\sup}(z) = g_{\Omega_0}(z) > 0$ for all $z \in \Omega = \Omega_0$.*

(iii) *$\operatorname{cap}(\partial\Omega_0) > \kappa_{\inf} > 0$. Here, $0 < g_{\sup}(z) < \infty$ for all $z \in \Omega \backslash \{\infty\}$, and $g_{\sup}(z) > g_{\Omega_0}(z)$ for all $z \in \Omega_0 \backslash \{\infty\}$.*

Alternatives (ii) and (iii) of Theorem 2.10(c), (d) occurred in Example 2.9. The alternative (i) will be illustrated in Example 5.2 below, here both cases $\operatorname{cap}(\partial\Omega_0) = 0$ and $\operatorname{cap}(\partial\Omega_0) > 0$ are possible.

In the proof of Theorem 2.10, we will make use of

LEMMA 2.11. (a) *For any $g \in \mathscr{G}$ there hold $g(z) \geqslant g_{\Omega_0}(z)$ for $z \in \omega_0$, with equality for one z if and only if $g = g_{\Omega_0}$ in Ω_0. Furthermore, for the constant κ_Λ as defined in Theorem 2.5(b) there holds $\kappa_\Lambda \leqslant \operatorname{cap}(\partial\Omega_0)$, with equality if and only if $g = g_{\Omega_0}$ in Ω_0.*

(b) *We have the equivalences $\kappa_{\sup} = 0 \Leftrightarrow \mathscr{G} = \{+\infty\}$, and $\kappa_{\inf} = 0 \Leftrightarrow +\infty \in \mathscr{G}$.*

Proof. Let D be some domain containing ∞, with its closure F being a subset of Ω_0, and $\text{cap}(\partial F) > 0$; furthermore, suppose that D is regular with respect to the Dirichlet problem. From Theorem 2.5(b), (c), we know that $f_D := g - g_D$ is harmonic and continuous on F, and $f_D(z) = g(z) - g_D(z) = g(z) \geqslant g_{\text{inf}}(z) > 0$ for $z \in \partial D$. Thus, by the maximum principle for harmonic functions, there holds $g(z) \geqslant g_D(z)$ for all $z \in F$. Since g_{Ω_0} may be defined by an exhaustion of Ω_0 as the local uniform limit of Green functions g_D of the above type (see, e.g., [16, Chap. V.5.3]), it follows in the case $\text{cap}(\partial\Omega) > 0$ that $f := g - g_{\Omega_0}$ is harmonic and nonnegative in Ω_0, with $f(\infty) = \log(\text{cap}(\partial\Omega_0)/\kappa_\Lambda)$. Furthermore, by the minimum principle for harmonic functions we have either $f = 0$ in Ω_0 or $f > 0$ in Ω_0 (including infinity), as claimed in part (a). In the case $\text{cap}(\partial\Omega) = 0$ there holds $g_{\Omega_0} = +\infty$ in Ω_0. Here, taking the above limit, we get $\kappa_\Lambda = 0$ and $g = \infty$ in Ω_0, and again assertion (a) follows.

For a proof of (b), notice that $\kappa_{\text{sup}} = 0$ (and $\kappa_{\text{inf}} = 0$, respectively) iff $\kappa_\Lambda = 0$ in Theorem 2.5(b) for any infinite set $\Lambda \subset \mathbb{N}$ (for some infinite sets $\Lambda \subset \mathbb{N}$ according to the normality established in Theorem 2.5(a)), showing the above equivalences. ∎

Proof of Theorem 2.10(a). Given a $\zeta \in \Omega_0 \setminus \{\infty\}$, according to (22), we may find $g \in \mathcal{G}$ with $g(\zeta) = g_{\text{inf}}(\zeta)$. In the case $\kappa_{\text{sup}} = 0$, the assertion is trivial since $g = g_{\text{inf}} = +\infty$ by Lemma 2.11(b). Therefore, suppose that $\kappa_{\text{sup}} > 0$, and consider the function $f(z) := g(z) - g_{\text{inf}}(z)$. From Theorem 2.5(b), (c), and (22), we know that f is subharmonic, non-negative and continuous in Ω_0, with $f(\infty) = 0$ by construction. Taking account of the minimum principle for harmonic functions, we may conclude that f is harmonic in Ω_0 iff $f = 0$ in Ω_0, as claimed in Theorem 2.10(a). A proof for g_{sup} and for other connected components of Ω is similar, we omit the details. ∎

Proof of Theorem 2.10(b) (iii) $\Rightarrow$ (ii), (i). If g_{inf} coincides with g_{Ω_0} in Ω_0 then in particular it is harmonic in $\Omega_0 \setminus \{\infty\}$. From part (a) it follows that there exists a $g \in \mathcal{G}$ coinciding with g_{Ω_0} in Ω_0. Taking into account that the Green function g_{Ω_0} has limit boundary values 0 quasi everywhere on $\partial\Omega_0$, we have established the implications as claimed above.

(i) $\Rightarrow$ (ii). Let g have limit boundary values equal to zero quasi everywhere on $\partial\Omega_0$. From (22), we know that $0 < g_{\text{inf}} \leqslant g$ in Ω_0, and thus (ii) is trivially true.

(ii) $\Rightarrow$ (iii). From Lemma 2.11(b), we know that $\kappa_{\text{sup}} = 0$ implies that $g_{\text{inf}} = +\infty$ in Ω_0. Here, (ii) implies that $\text{cap}(\partial\Omega_0) = 0$, and the assertion (iii) is true. It remains to discuss the case $\kappa_{\text{sup}} > 0$, and thus $\text{cap}(\partial\Omega_0) > 0$ by Lemma 2.11(a). According to the fact that $\partial\Omega_0 \subset \sigma(A) \subset \{z \in \mathbb{C}\colon |z| \leqslant \|A\|\}$, one verifies that $g_{\Omega_0}(z) \leqslant \log(1/\text{cap}(\partial\Omega_0)) + \log(|z| + \|A\|)$ for $z \in \mathbb{C}$.

Consequently, the function $f := g_{\inf} - g_{\Omega_0}$ is bounded below in Ω_0, superharmonic in Ω_0, with limit boundary values equal to 0 quasi everywhere on $\partial\Omega_0$. From the second maximum principle (see, e.g., [20, p. 223]), we may conclude that $g_{\inf} \leqslant g_{\Omega_0}$ in Ω_0. A combination with Lemma 2.11(a) and (22) gives $g_{\inf} = g_{\Omega_0}$ in Ω_0, as claimed in (iii).

It remains to deduce that $\Omega = \Omega_0$. Denote by F the (compact) complement of Ω_0, and by D the interior of F. Given g as in part (i), we choose Λ and κ_Λ as in Theorem 2.5(b). As above, it follows from the second maximum principle that $g = g_{\Omega_0}$ in Ω_0; in particular, $\kappa_\Lambda = \operatorname{cap}(\partial\Omega_0) = \operatorname{cap}(F)$. Notice that $\operatorname{cap}(F) = 0$ implies that the interior of F is empty, and thus $\Omega = \Omega_0$. Therefore, let $\operatorname{cap}(F) > 0$. By possibly taking a subsequence, we may assume in addition that the sequences of zero counting measures $(\mu_n)_{n \in \Lambda}$, and $(\mu_{n+1})_{n \in \Lambda}$ converge weakly to some unit measures μ, and μ', respectively. Recall from Proposition 2.1 that μ, μ' are supported on F. Also, in Lemma 2.7 we have shown that the potentials of both measures μ', μ coincide in Ω_0, and thus by (iii)

$$
\begin{aligned}
g_{\Omega_0}(z) = g(z) &= \log \frac{1}{\operatorname{cap}(F)} - V[\mu](z) \\
&= \log \frac{1}{\operatorname{cap}(F)} - V[\mu'](z), \qquad z \in \Omega_0.
\end{aligned}
\tag{25}
$$

From the continuity in the fine topology (see [20, Appendix A.2] and the argument used in [20, Proof of Theorem 2.2.1(b) and Proof of Theorem 3.1.1]), we may conclude that (25) is also valid for $z \in \partial\Omega_0 = \partial F$. In particular, both potentials $V[\mu]$ and $V[\mu']$ are equal to $\log(1/\operatorname{cap}(F))$ quasi everywhere in ∂F. By the second maximum principle, we get

$$
V[\mu](z) \geqslant \log \frac{1}{\operatorname{cap}(F)}, \qquad V[\mu'](z) \geqslant \log \frac{1}{\operatorname{cap}(F)}, \qquad z \in D.
$$

Using the principle of descent (see, e.g., [20, p. 222]), we may conclude that, for $z \in D$,

$$
\begin{aligned}
&\lim_{n \to \infty,\, n \in \Lambda} \log |v_n(z)|^{1/n} \\
&\quad = \log \frac{1}{\operatorname{cap}(F)} + \lim_{n \to \infty,\, n \in \Lambda} \max\{-V[\mu_n](z), -V[\mu_{n+1}](z)\} \\
&\quad \leqslant \log \frac{1}{\operatorname{cap}(F)} + \max\{-V[\mu](z), -V[\mu'](z)\} \leqslant 0,
\end{aligned}
$$

and thus $z \notin \Omega$, showing that $\Omega = \Omega_0$. ∎

Proof of Theorem 2.10(c), (d). Notice first that by Lemma 2.11(a) there holds $0 \leqslant \kappa_{\inf} \leqslant \kappa_{\sup} \leqslant \mathrm{cap}(\partial\Omega_0)$. If alternative (c)(i) or (d)(i) holds, then the corresponding assertions follow from Lemma 2.11(b) together with (22), (23). Otherwise, it follows from Lemma 2.11(a) that $\mathrm{cap}(\partial\Omega) > 0$.

In cases (c)(ii), (c)(iii), the inequalities $0 < g_{\inf}(z) < \infty$ for $z \in \Omega \backslash \{\infty\}$ are a consequence of Theorem 2.5(b), (c) and (22). The inequality $g_{\inf}(z) \geqslant g_{\Omega_0}(z)$, for $z \in \Omega_0$, follows from Lemma 2.11(a) and (22). Now, if $g_{\inf}(\zeta) = g_{\Omega_0}(\zeta)$ for some $\zeta \in \Omega_0$, then by (22) there exists an $g \in \mathcal{G}$ with $g(\zeta) = g_{\Omega_0}(\zeta)$. It follows from Lemma 2.11(a) that $g = g_{\Omega_0}$ in Ω_0, and thus $g_{\inf} = g_{\Omega_0}$ in Ω_0 and in particular $\Omega = \Omega_0$ by part (b). Furthermore, we see from the second part of Lemma 2.11(a) that this case occurs if and only if $\kappa_{\sup} = \mathrm{cap}(\partial\Omega_0)$. This establishes the assertions of alternatives (c)(ii) and (c)(iii).

In cases (d)(ii) and (d)(iii), we use (23) and the above arguments to show that $0 < g_{\sup}(z) < \infty$ for $z \in \Omega \backslash \{\infty\}$ and $g_{\sup}(z) \geqslant g_{\Omega_0}(z)$ for $z \in \Omega_0$. By part (c) and (23), equality holds for some ζ in the second estimate iff $g(\zeta) = g_{\Omega_0}(\zeta)$ for all $g \in \mathcal{G}$. Thus a proof for the remaining assertions of alternatives (d)(ii) and (d)(iii) follows closely that of part (c), we omit the details. ∎

To conclude this subsection, we characterize a subclass of operators with formal orthonormal polynomials having *regular asymptotic behavior*, generalizing the concept of orthogonality with respect to a measure supported on $[-1, 1]$ (see [22]) or on a compact subset of the real line (compare [20, Sect. 3]).

COROLLARY 2.12. *The following three statements are equivalent*:

(a) $g_{\sup} = g_{\Omega_0}$ *in* Ω_0, *i.e.*, $(v_n^{-1/n})_{n \geqslant 0}$ *converges locally uniformly in* Ω_0 *to* $\exp(-g_{\Omega_0}(\cdot))$.

(b) $\kappa_{\inf} = \mathrm{cap}(\partial\Omega_0)$, *i.e.*, *the sequence* $(|k_n|^{-1/n})_{n \geqslant 0}$ *converges to* $\mathrm{cap}(\partial\Omega_0)$.

(c) *The limit* $\limsup_{n \to \infty} |q_n(z)|^{1/n} \leqslant 1$ *holds true quasi everywhere on* $\partial\Omega_0$.

In any of these cases, we have

(d) $\Omega = \Omega_0$ *is connected.*

(e) *If in addition* $\sigma(A)$ *has empty interior and* $\mathrm{cap}(\sigma(A)) > 0$, *then the sequence of zero counting measures of* q_n, $n \geqslant 0$, *converges weakly to the equilibrium measure of* $\sigma(A)$.

Proof. (b) $\Rightarrow$ (a), (d). From Theorem 2.10(c), (d), we know that assertion (b) implies $g_{\sup} = g_{\inf} = g_{\Omega_0}$ in Ω_0, and thus part (d) by Theorem 2.10(b). In particular, we obtain $\mathcal{G} = \{g_{\Omega_0}\}$, and the convergence claimed in part (a) follows from Theorem 2.5(a).

(a)$\Rightarrow$(b). From part (a) together with the inequalities $g_{\sup} \geqslant g_{\inf} \geqslant g_{\Omega_0}$ in Ω_0, we may conclude that $g_{\sup} = g_{\inf} = g_{\Omega_0}$ in Ω_0, and thus $\Omega_0 = \Omega$ by Theorem 2.10(b). Consequently, the assertion of part (b) follows from Theorem 2.10(c), (d).

(c)$\Rightarrow$(b). Notice first that in the case $\mathrm{cap}(\partial\Omega_0) = 0$ the assertion (b) trivially follows from Theorem 2.10(d). Also, from Theorem 2.10(d), we know that $\mathrm{cap}(\partial\Omega_0) \geqslant \kappa_{\inf}$, and it remains to establish the reverse inequality in the case $\mathrm{cap}(\partial\Omega_0) > 0$. Here, we use arguments similar to [20, Proof of Theorem 3.1.1.]: let $\Lambda \subset \mathbb{N}$ be such that $(|k_n|^{-1/n})_{n\in\Lambda}$ converges to κ_Λ. By taking a subsequence, we may suppose in addition that $(\mu_n)_{n\in\Lambda}$ converges weakly to some measure μ_Λ. By assumption,

$$0 \geqslant \limsup_{n\to\infty,\, n\in\Lambda} \log |q_n(z)|^{1/n} = \limsup_{n\to\infty,\, n\in\Lambda} \log \frac{1}{|k_n|^{-1/n}} - V[\mu_n](z)$$

$$= \log \frac{1}{\kappa_\Lambda} - \liminf_{n\to\infty,\, n\in\Lambda} V[\mu_n](z)$$

for quasi all $z \in \partial\Omega_0$. From the lower envelope theorem [20, p. 223] it follows that

$$\log \frac{1}{\kappa_\Lambda} - V[\mu_\Lambda](z) \leqslant 0 \qquad \text{for quasi all} \quad z \in \partial\Omega_0. \tag{26}$$

In particular, since $\mathrm{cap}\{z \in \mathbb{C}\colon V[\mu](z) = +\infty\} = 0$ for any measure μ (see, e.g., [20, p. 222]), we obtain $\kappa_\Lambda > 0$. Since $\mathrm{cap}(\partial\Omega_0) > 0$, the equilibrium measure ω of $\mathbb{C}\backslash\Omega_0$ exists and is of finite energy. Notice that the exceptional set $E := \{z \in \partial\Omega_0\colon \log(1/\kappa_\Lambda) - V[\mu_\Lambda](z) > 0\}$ has capacity zero according to (26), and thus $\omega(E) = 0$. Taking into account that ω is supported on $\partial\Omega_0$, we obtain

$$0 \geqslant \int \left[\log \frac{1}{\kappa_\Lambda} - V[\mu_\Lambda](z) \right] d\omega(z) = \log \frac{1}{\kappa_\Lambda} - \int V[\mu_\Lambda](z)\, \omega(z)$$

$$= \log \frac{1}{\kappa_\Lambda} - \int V[\omega](z)\, d\mu_\Lambda(z) = \log \frac{\mathrm{cap}(\partial\Omega_0)}{\kappa_\Lambda} + \int g_{\Omega_0}(z)\, d\mu_\Lambda(z),$$

where we have used the fact that $g_{\Omega_0}(z) = \log(1/\mathrm{cap}(\partial\Omega_0)) - V[\omega](z)$ for $z \in \mathbb{C}$ (see, e.g., [20, p. 227]). Notice that the integral on the right hand side is nonnegative, showing that $\kappa_\Lambda \geqslant \mathrm{cap}(\partial\Omega_0)$ for any such set Λ, and in particular $\kappa_{\inf} \geqslant \mathrm{cap}(\partial\Omega_0)$.

(a), (b)$\Rightarrow$(c). If $\mathrm{cap}(\partial\Omega_0) = 0$ then (c) trivially holds. Therefore, let $\mathrm{cap}(\partial\Omega_0) > 0$, and denote again the equilibrium measure of $\partial\Omega$ by ω. As a first step let us show that $V[\omega](z) = V[\mu](z)$ for $z \in \partial\Omega_0$ for any weak accumulation point μ of $(\mu_n)_{n\geqslant 0}$. Let $\Lambda\subset\mathbb{N}$ be such that $(\mu_n)_{n\in\Lambda}$ converge weakly to μ. By part (b), the sequence $(|k_n|^{1/n})_{n\geqslant 0}$ converges to $1/\mathrm{cap}(\partial\Omega_0)$. Thus, from part (a) and Theorem 2.5, we obtain the representation $g_{\Omega_0}(z) = \log(1/\mathrm{cap}(\partial\Omega_0)) - V[\mu](z)$ for $z\in\Omega_0$, and thus $V[\omega](z) = V[\mu](z)$ for $z\in\Omega_0$. As before, we conclude that this equality remains valid on $\partial\Omega$, leading to the claim above. For establishing (c), it remains to show that the set

$$E=\{z\in\partial\Omega\colon \limsup_{n\to\infty} \log|q_n(z)|^{1/n}>0\}$$

is of capacity zero. Since $(|k_n|^{-1/n})_{n\geqslant 0}$ converges to $\mathrm{cap}(\partial\Omega_0)>0$, we have

$$E=\left\{z\in\partial\Omega\colon \liminf_{n\to\infty} V[\mu_n](z)<\log\frac{1}{\mathrm{cap}(\partial\Omega_0)}\right\}.$$

By the principle of descent (see, e.g., [20, p. 222]), it follows that

$$E\subset E'=\left\{z\in\partial\Omega\colon V[\mu](z)<\log\frac{1}{\mathrm{cap}(\partial\Omega_0)}\right.$$
$$\left.\text{for some weak accumulation point } \mu \text{ of } (\mu_n)_n\right\}.$$

Taking into account the assertion above, we obtain the description

$$E'=\{z\in\partial\Omega\colon V[\omega](z)<\log(1/\mathrm{cap}(\partial\Omega_0))\}=\{z\in\partial\Omega\colon g_{\Omega_0}(z)>0\}.$$

Consequently, $\mathrm{cap}(E')=0$ and thus $\mathrm{cap}(E)=0$, showing (c).

(a), (b) $\Rightarrow$ (e). First, note that parts (a), (b) together with Theorem 2.5(b) imply that for any weak accumulation point μ_Λ of $(\mu_n)_{n\geqslant 0}$ there holds $\log(1/\mathrm{cap}(\partial\Omega_0)) - V[\mu_\Lambda](z) = g_{\Omega_0}(z)$ for $z\in\Omega_0$, and thus also for $z\in\partial\Omega_0$. Now, if the interior of $\sigma(A)$ is empty, then $\sigma(A)=\mathbb{C}\backslash\Omega_0=\partial\Omega_0$. It follows that this identity holds everywhere in $\mathbb{C}$, and thus μ_Λ coincides with the equilibrium measure of $\sigma(A)$. ∎

If one of the conditions (a)–(c) of Corollary 2.12 is satisfied, we will write[3] for short $A\in\mathbf{Reg}$. If in addition $\sigma(A)$ has empty interior and $\mathrm{cap}(\sigma(A))>0$ then in terms of the remark before Example 2.9 it follows that A has a density-of-states measure which coincides with the equilibrium measure of $\sigma(A)$. Note however that the reciprocal of this assertion is not

[3] The class **Reg** of particular finite Borel measures with (possibly complex) compact support has been introduced by Stahl and Totik [20]. It follows from [20, Theorem 3.1.1(a)] that, for the particular case of bounded self-adjoint operators A, we have $A\in\mathbf{Reg}$ iff its spectral measure is of class **Reg**.

true (i.e., (e) does not imply $A \in \mathbf{Reg}$). To see this, take in Example 5.2 below as set E some subset of the real axis with positive capacity, and as $(b_n)_{n \geqslant 0}$ for instance the sequence of *Leja points* of E. Then the density-of-state measure of $\tilde{A}$ is shown to be the equilibrium measure of E, and the same is true for A by [11, Theorem 2.4], though $\kappa_{\text{inf}} = \kappa_{\text{sup}} = 0 \neq \operatorname{cap}(\partial\Omega_0)$.

From results given in [8, Sects. 2.2 and 2.3] it follows that operators with complex periodic Jacobi matrices or more generally operators with complex asymptotically periodic Jacobi matrices are elements of **Reg**.

3. CONVERGENCE OF BOUNDED *J*-FRACTIONS IN CAPACITY OR MEASURE

As our first convergence result for a general bounded *J*-fraction, we may now show that there is convergence in capacity of the whole sequence of Padé approximations in the outer domain Ω_0 of the resolvent set of the corresponding second order difference operator. Note that the restriction to the outer domain is natural, since the Weyl function ϕ is approximated in terms of its Laurent series at infinity; moreover, according to the special case of self-adjoint operators, we may not expect that a sharper form of convergence is valid in the whole outer domain Ω_0.

THEOREM 3.1. *The sequence of Padé approximants $(\pi_n)_{n \geqslant 0}$ converges in capacity in Ω_0 to the Weyl function. More precisely, for any closed set $F \subset \Omega_0$ and for any $\varepsilon > 0$ there holds*

$$\lim_{n \to \infty} \operatorname{cap}\{z \in F \colon |\phi(z) - \pi_n(z)|^{1/(2n)} \geqslant e^{-g_{\inf}(z)} + \varepsilon\} = 0, \tag{27}$$

$$\lim_{n \to \infty} \operatorname{cap}\{z \in F \colon |\phi(z) - \pi_n(z)|^{1/(2n)} \leqslant e^{-g_{\sup}(z)} - \varepsilon\} = 0, \tag{28}$$

where $\operatorname{cap}(\cdot)$ *denotes the logarithmic capacity.*

Proof. Let $F \subset \Omega_0$ be closed and bounded,[4] and $\varepsilon > 0$. Furthermore, let $\omega \in \sigma(A)$ be some fixed complex number. First recall from the proof of Proposition 2.1 that the functions $f_n(z) := (z - \omega) \cdot r_n(z) \cdot q_n(z)$, $n \geqslant 0$, $z \in \Omega_0 =: D$ meet the requirements for Lemma 2.4(c). Thus there exist positive constants C, v and monic polynomials $\hat{f}_n$, $n \geqslant 0$, of degree $v_n \leqslant v$ such that

$$C \cdot |\hat{f}_n(z)| \leqslant |f_n(z)|, \qquad n \geqslant 0, \quad z \in F.$$

[4] By combining the last part of Proposition 2.1 with the first part of Lemma 2.4(c), we see that the two estimates (30) below remain valid with $\hat{f}_n = 1$ for z lying in some closed neighborhood U of infinity. Thus it is sufficient to prove Theorem 3.1 for compact sets F.

Taking into account (14) and Lemma 2.3, we get for $z \in F$ and $n \geqslant 0$

$$|\phi(z)-\pi_n(z)|^{1/2n} = \left(\frac{|(z-\omega)\cdot r_n(z)^2|}{|f_n(z)|}\right)^{1/2n} \leqslant \frac{(|z-\omega|\cdot(1+\|A\|^2)\cdot\beta(z)^2/C)^{1/2n}}{|\hat{f}_n(z)|^{1/2n}\cdot v_{n-1}(z)^{1/n}},$$

$$|\phi(z)-\pi_n(z)|^{1/2n} = \left(\frac{|f_n(z)|}{|(z-\omega)\cdot q_n(z)^2|}\right)^{1/2n} \geqslant \frac{(C/\mathrm{dist}(\omega, F))^{1/2n}\cdot|\hat{f}_n(z)|^{1/2n}}{v_n(z)^{1/n}}. \tag{29}$$

Since $|z|\cdot\beta(z)$ is continuous in Ω, it follows from Theorem 2.5(c) that there exists an $N=N(F,\varepsilon)$ such that for all $n\geqslant N$ and for all $z\in F$

$$\begin{aligned} |\phi(z)-\pi_n(z)|^{1/2n}\cdot|\hat{f}_n(z)|^{1/2n} &\leqslant \exp(-g_{\inf}(z))+\frac{\varepsilon}{2},\\ |\phi(z)-\pi_n(z)|^{1/2n}/|\hat{f}_n(z)|^{1/2n} &\geqslant \exp(-g_{\sup}(z))-\frac{\varepsilon}{2}. \end{aligned} \tag{30}$$

We are now prepared to show (27). In fact, the exceptional set $\{z\in F\colon |\phi(z)-\pi_n(z)|^{1/2n}\geqslant \exp(-g_{\inf}(z))+\varepsilon\}$ is included in $\{z\in F\colon |\hat{f}_n(z)|^{1/2n}\leqslant (\exp(-g_{\inf}(z))+\varepsilon/2)/(\exp(-g_{\inf}(z))+\varepsilon)\}$ for $n\geqslant N$, which is a subset of

$$U_n := \left\{z\in\mathbb{C}\colon |\hat{f}_n(z)|^{1/2n}\leqslant\frac{1}{\eta}\right\}, \qquad \eta := \frac{1+\varepsilon}{1+\varepsilon/2}>1,$$

since $g_{\inf}$ is positive on F. According to the monotoniticy of the set function $\mathrm{cap}(\cdot)$, for the assertion (27) of Theorem 3.1 it is sufficient to show that $(\mathrm{cap}(U_n))_{n\geqslant 0}$ tends to zero.

Let $g_n(z) := \log([\eta\cdot|\hat{f}_n(z)|^{1/2n}]^{2n/v_n})$, then g_n is nonnegative and harmonic in $\mathbb{C}\backslash U_n$, zero on the boundary of U_n, and

$$g_n(z) = \log|z| + \log\frac{1}{\eta^{-2n/v_n}} + o(1)_{|z|\to\infty}.$$

Thus g_n is the Green function of U_n, and $\mathrm{cap}(U_n)=\eta^{-2n/v_n}\leqslant\eta^{-2n/v}$, which for $n\to\infty$ tends to zero.

It remains to show (28) which trivially holds if $g_{\sup}$ equals the constant $+\infty$ in Ω_0. Otherwise, we know from Theorem 2.5(c) that $g_{\sup}$ is positive and continuous on the compact set F, and by possibly making ε smaller, we may insure that $\exp(-g_{\sup}(z))\geqslant 3\varepsilon/2$ for all $z\in F$. It follows from (30) that the exceptional set $\{z \in F\colon |\phi(z) - \pi_n(z)|^{1/(2n)} \leqslant \exp(-g_{\sup}(z)) - \varepsilon\}$ is a subset of $\{z \in F\colon |\hat{f}_n(z)|^{1/2n} \leqslant (\exp(-g_{\sup}(z)) - \varepsilon)/(\exp(-g_{\sup}(z)) - \varepsilon/2)\}$ for $n \geqslant N$. Therefore we may estimate the capacity of the exceptional set as in the first part of the proof. ∎

For a class of (multivalued) functions ϕ including algebraic functions, Stahl [18, Theorem 2] established convergence in capacity of the whole sequence of Padé approximants in a domain D with the rate $\exp(-g_D(z))$. This set D is uniquely characterized by the fact that the compact set $\mathbb{C} \backslash D$ is the (smallest) set of minimal capacity outside of which ϕ is a single-valued analytic function. Of course, algebraic functions do not necessarily have a (bounded) *J*-fraction expansion around infinity. However, for such a subclass, we get from Theorem 3.1 that $g_{\sup} = g_{\inf} = g_D$, and $\Omega_0 \subset D$. Also, at least for some special cases (namely periodic difference operators, see [8, Remark 2.9]), we know that $\Omega_0 = D$, leading to a quite interesting description of the resolvent set of a second order difference operator in terms of analytic continuation of its Weyl function ϕ. A description of spectral properties of such an operator in terms of analytic properties of its Weyl function will also be the subject of subsequent considerations in Section 5.

In the next assertions, we study the question whether the rate of convergence given in Theorem 3.1 is the best possible.

THEOREM 3.2. (a) *In Theorem* 3.1, *the function* $\exp(-g_{\inf}(\cdot))$ *may not be replaced by any smaller function continuous in* Ω_0. *More precisely, for any* ε *and for any* $\zeta \in \Omega$ *there exist a neighborhood* $U \subset \Omega$ *of* ζ *and an infinite set* $\Lambda \subset \mathbb{N}$ *such that*

$$\liminf_{n \to \infty,\, n \in \Lambda} |\phi(z) - \pi_n(z)|^{1/(2n)} \geqslant e^{-g_{\inf}(z)} - \varepsilon, \qquad z \in U. \tag{31}$$

Similarly, the function $\exp(-g_{\sup}(\cdot))$ *may not be replaced by any larger function continuous in* Ω_0.

(b) *There exists an infinite set* $\Lambda \subset \mathbb{N}$ *such that, for any compact set* $F \subset \Omega_0$ *and for any* $\varepsilon > 0$,

$$\lim_{n \to \infty,\, n \in \Lambda} \operatorname{cap}\{z \in F\colon |\phi(z) - \pi_n(z)|^{1/(2n)} \leqslant e^{-g_{\inf}(z)} - \varepsilon\} = 0$$

if and only if $g_{\inf}$ *is harmonic in* $\Omega_0 \backslash \{\infty\}$ (*including the constant* $+\infty$). *A similar remark holds for* $g_{\sup}$.

For the proof we require a property which is also of independent interest.

LEMMA 3.3. *For any $\zeta \in \Omega$ there exists a neighborhood $U \subset \Omega$ of ζ and integers $\varepsilon_n \in \{0, 1\}$, $n \geqslant 0$, such that*

$$|a_n \cdot r_{n+\varepsilon_n}(z) \cdot q_{n+1-\varepsilon_n}(z)| \geqslant 1/4, \qquad n \geqslant 0, \quad z \in U. \tag{32}$$

Proof. By (15) and the Montel Theorem, the sequences of functions $(a_n \cdot r_n \cdot q_{n+1})_{n \geqslant 0}$ and $(a_n \cdot r_{n+1} \cdot q_n)_{n \geqslant 0}$ of functions analytic in Ω are normal in Ω, and in particular equicontinuous. According to (19), we find $\varepsilon_n \in \{0, 1\}$ such that

$$|a_n \cdot r_{n+\varepsilon_n}(\zeta) \cdot q_{n+1-\varepsilon_n}(\zeta)| \geqslant 1/2, \qquad n \geqslant 0.$$

Thus (32) follows by equicontinuity by taking a sufficiently small neighborhood U. ∎

Proof of Proposition 3.2(a). By definition (9), there exists a Λ' such that

$$\lim_{n \to \infty,\, n \in \Lambda'} v_n(\zeta)^{-1/n} = e^{-g_{\inf}(\zeta)}.$$

In Theorem 2.5(a) we have shown that $(v_n^{-1/n})_{n \in \Lambda'}$ is equicontinuous in Ω, and the continuity of $\exp(-g_{\inf}(\cdot))$ in Ω was established in Theorem 2.5(c). Thus, there exist a neighborhood U of ζ and some N such that

$$|v_n(z)^{-1/n} - \exp(-g_{\inf}(z))| < \varepsilon/2$$

for $z \in U$ and for $n \in \Lambda'$, $n \geqslant N$. By possibly choosing a smaller U, we get with the help of Lemma 3.3

$$|\phi(z) - \pi_{n+\varepsilon_n}(z)|^{1/(2n)} \geqslant \frac{1}{|4 \cdot a_n \cdot q_n(z) \cdot q_{n+1}(z)|^{1/(2n)}} \geqslant \frac{v_n(z)^{-1/n}}{2^{1/2n}},$$

$n \geqslant 0$, $z \in U$. Combining both estimates, we see that the set $\Lambda := \{n + \varepsilon_n : n \in \Lambda'\}$ has the required properties. ∎

Proof of Proposition 3.2(b). Suppose first that $g_{\inf}$ is harmonic in $\Omega_0 \backslash \{\infty\}$, and thus $g_{\inf}$ coincides in Ω_0 with some $g \in \mathcal{G}$ by Theorem 2.10(a). We choose the corresponding set Λ as in Theorem 2.5(b). Furthermore, let $F \subset \Omega_0$ be compact, and $\varepsilon > 0$ sufficiently small. With the notation of the proof of Theorem 3.1, it follows from (29) that there is an N such that

$$|\phi(z) - \pi_n(z)|^{1/2n} / |\hat{f}_n(z)|^{1/2n} \geqslant \exp(-g_{\inf}(z)) - \frac{\varepsilon}{2}$$

for $n \in \Lambda$, $n \geqslant N$, and for $z \in F$. Consequently, the estimate for the capacity of the exceptional set may be obtained as in the second part of the proof of Theorem 3.1.

In order to show the other implication, we will apply Theorem 2.5(a): by possibly taking a subsequence, we may assume that $(v_n^{-1/n})_{n+1 \in \Lambda}$ converges locally uniformly in Ω to some function $v = \exp(-g)$ for some $g \in \mathcal{G}$. According to Theorem 2.10(a), it is sufficient to show that $v = \exp(-g_{\inf})$ in Ω_0. Suppose the contrary, i.e., let $\zeta \in \Omega_0 \setminus \{\infty\}$ with $\exp(-g_{\inf}(\zeta)) - v(\zeta) =: 3\varepsilon > 0$. By continuity, we find some compact neighborhood F of ζ such that $\exp(-g_{\inf}(z)) - v(z) \geqslant 2\varepsilon$ for $z \in F$. Then as in the proof of Theorem 3.1 one shows that

$$\lim_{n \to \infty,\, n \in \Lambda} \operatorname{cap}\{z \in F\colon |\phi(z) - \pi_n(z)|^{1/(2n)} \geqslant v(z) + \varepsilon\} = 0.$$

Consequently,

$$\lim_{n \to \infty,\, n \in \Lambda} \operatorname{cap}\{z \in F\colon |\phi(z) - \pi_n(z)|^{1/(2n)} \geqslant e^{-g_{\inf}(z)} - \varepsilon\} = 0,$$

which obviously contradicts the hypothesis of Proposition 3.2(b). ∎

By slightly modifying the proof of Theorem 3.1, we may establish convergence μ-almost everywhere for measures μ satisfying some regularity property, such as the (two-dimensional) Lebesgue measure or the α-dimensional Hausdorff measure [21].

Corollary 3.4. *Let μ be a positive measure satisfying $\mu(\Delta_r) \leqslant C \cdot r^\alpha$ for any closed disc Δ_r of radius $r > 0$, where α, C are some positive constants. Then $(\pi_n)_{n \geqslant 0}$ converges μ-almost everywhere in Ω_0 to the Weyl function ϕ, and*

$$\limsup_{n \to \infty} |\phi(z) - \pi_n(z)|^{1/(2n)} = g^{-g_{\inf}(z)},$$

$$\liminf_{n \to \infty} |\phi(z) - \pi_n(z)|^{1/(2n)} = e^{-g_{\sup}(z)}$$

for μ-almost all $z \in \Omega_0$.

Proof. From Proposition 3.2(a), we know that

$$\limsup_{n \to \infty} |\phi(z) - \pi_n(z)|^{1/(2n)} \geqslant e^{-g_{\inf}(z)}$$

and

$$\liminf_{n \to \infty} |\phi(z) - \pi_n(z)|^{1/(2n)} \leqslant e^{-g_{\sup}(z)}$$

for all $z \in \Omega$. Thus for the assertion of Corollary 3.4 it is sufficient to show that for any closed set $F \subset \Omega_0$ and for any $\varepsilon > 0$ there holds

$$\lim_{m \to \infty} \mu \left(\bigcup_{n=m}^{\infty} \{ z \in F \colon |\phi(z) - \pi_n(z)|^{1/(2n)} \geqslant e^{-g_{\inf}(z)} + \varepsilon \} \right) = 0,$$

and that a similar estimate is true for the union of exceptional sets for the upper bound $e^{-g_{\sup}(z)} - \varepsilon$. With the notations of the proof of Theorem 3.1, this follows by establishing that

$$\lim_{n \to \infty} \mu \left(\bigcup_{n=m}^{\infty} U_n \right) = 0, \qquad U_n = \left\{ z \in \mathbb{C} \colon |\hat{f}_n(z)|^{1/2n} \leqslant \frac{1}{\eta} \right\}, \qquad n \geqslant 0,$$

where $\hat{f}_n$, $n \geqslant 0$, is some monic polynomial of degree bounded by v. Denote by $\tilde{U}_n$, $n \geqslant 0$, the union of circles of radius $r_n := \eta^{-2n/v} < 1$, centered at the zeros of $\hat{f}_n$. Then

$$|\hat{f}_n(z)| \geqslant r_n^{\deg \hat{f}_n} \geqslant r_n^v = \frac{1}{\eta^{2n}}, \qquad z \notin \tilde{U}_n,$$

implying that $U_n \subset \tilde{U}_n$ for $n \geqslant 0$. Consequently,

$$\mu \left(\bigcup_{n=m}^{\infty} U_n \right) \leqslant \sum_{n=m}^{\infty} \mu(\tilde{U}_n) \leqslant C \cdot v \cdot \sum_{n=m}^{\infty} r_n^{\alpha} \leqslant C_1 \cdot \left(\frac{1}{\eta^{2\alpha/v}} \right)^m$$

for all $m \geqslant 0$, with C_1 some constant. Thus this expression tends to zero. ∎

For a summary of other results concerning convergence of Padé approximants in measure and/or capacity we refer to [3, Chaps. 6.5 and 6.6]. A number of open problems in this context are given by Stahl [19].

4. LOCAL UNIFORM CONVERGENCE OF BOUNDED *J*-FRACTIONS

In this section we study the question of uniform convergence in (some part of) the resolvent set.

In [2, Theorem 2], Aptekarev *et al.* showed that there is pointwise convergence of a subsequence of Padé approximants to the Weyl function in the whole resolvent set Ω. However, the choice of indices for this subsequence may depend on the point under consideration, as it becomes clear from the following formula which is a consequence of [8, Corollary 3.4]

$$\limsup_{n\to\infty} \min\{|\phi(z)-\pi_n(z)|^{1/(2n)}, |\phi(z)-\pi_{n+1}(z)|^{1/(2n)}\}$$
$$\leqslant e^{-g_{\inf}(z)}, \qquad z\in\Omega. \tag{33}$$

Obviously, the poles of the rational functions π_n, $n\geqslant 0$, serve as an obstacle for uniform convergence. Therefore, it will be useful to consider asymptotically polefree domains. A domain D will be called *asymptotically polefree with respect to some infinite set* $\Lambda\subset\mathbb{N}$ if for any closed $F\subset D$ there exist an $n(F)$ such that the functions π_n are analytic on F for all $n\in\Lambda$, $n>n(F)$.

Polefree domains with respect to $\mathbb{N}$ are given by several authors, and in general also local uniform convergence in such domains is established. For instance, there are the Cassini ovals [23, Corollary 4.1], the Worpitski set (see [24, Theorem V.26.2; 8, Sect. 3.1]), the set $|z|>\|A\|$ [24, Theorem V.26.3], or more generally the complement of the closure of the numerical range of A (see [8, Theorem 3.10]), namely

$$\overline{\mathbb{C}}\backslash\overline{\{(Ay, y)\colon y\in l^2, \|y\|=1\}}\subset\Omega_0.$$

Note that all these domains contain a neighborhood of infinity. From examples reported in [8], we also know that in general the closure of the numerical range is larger than the convex hull of the spectrum. In this context, let us also mention some important results of Barrios *et al.* on second order difference operators which are (possibly complex) compact perturbations of a (possibly unbounded) self-adjoint operator [6] (for special cases see also [4, 5]): the authors prove that here the whole sequence of Padé approximants converges locally uniformly in $\Omega\backslash\mathbb{R}$.

Gonchar showed in [12, Theorem 1] that there is local uniform convergence of the whole sequence of Padé approximants in any asymptotically polefree domain D with respect to $\mathbb{N}$ containing infinity and satisfying an additional regularity property (namely, up to a set of capacity zero, the boundary ∂D has to be a subset of the boundary of the convex hull of the complement of D). On the other hand, such a property does not necessarily remain valid for subsequences (see [12, Sect. 3.2]).

The aim of the following theorem is to show that for asymptotically polefree subdomains of Ω_0 we may drop additional regularity properties.[5]

[5] By a Lemma of Gonchar, convergence in capacity implies local uniform convergence in asymptotically polefree subdomains. Here, we prefer to give a direct simple proof.

THEOREM 4.1. *Let $D \subset \Omega_0$ be an asymptotically polefree domain with respect to some infinite set $\Lambda \subset \mathbb{N}$. Then*

$$\limsup_{n \to \infty,\, n \in \Lambda} |\phi(z) - \pi_n(z)|^{1/(2n)} \leqslant e^{-g_{\inf}(z)}$$

locally uniformly in D.

Proof. By Proposition 2.2, the sequence $(u_{n-1})_{n \in \Lambda}$ of rational functions is normal in Ω_0 with respect to the chordal metric, with limit functions different from the constant infinity. Let F be some closed subset of D, and D' some domain containing F, with its closure being a subset of D. Then u_{n-1} is analytic in D' for all (sufficiently large) $n \in \Lambda$. From Lemma 2.4(d) and the Theorem of Montel, we may conclude that there exist constants N, C such that

$$|u_{n-1}(z)| \leqslant C, \qquad z \in F, \quad n \geqslant N, \quad n \in \Lambda.$$

Also, by (15),

$$|\phi(z) - \pi_n(z)| = \left| \frac{a_{n-1} \cdot r_n(z)}{a_{n-1} \cdot q_n(z)} \right| \leqslant \frac{\gamma(z)}{|a_{n-1} \cdot q_n(z)|^2}$$
$$= \frac{\gamma(z) \cdot (1 + |u_{n-1}(z)|^2)}{v_{n-1}(z)^2}$$

for $z \in \Omega$. Thus the assertion of Theorem 4.1 follows from the above two estimates together with Theorem 2.5(c). ∎

For self-adjoint A, the convex hull of $\sigma(A)$ contains all Padé poles (see, e.g., [20, Lemma 1.1.3]), in fact, it coincides with the closure of the numerical range. Thus the Markov convergence theorem is a special case of Theorem 4.1. Two other special cases are discussed in

COROLLARY 4.2. *Suppose that $(a_{n-1})_{n \in \Lambda}$ tends to zero. Then $(\pi_n)_{n \in \Lambda}$ converges to ϕ locally uniformly in Ω_0 (and in Ω).*

Proof. Let $z \in \Omega$. Using (13), we get for $n \geqslant 1$

$$\begin{aligned}
\|(zI - A)^{-1}\|^2 \cdot (1 + |s_n(z)|^2) &= \|(zI - A)^{-1}\|^2 \cdot \|q_n(z) \cdot e_0 - e_n\|^2 \\
&\geqslant \|(zI - A)^{-1} (q_n(z) \cdot e_0 - e_n)\|^2 \\
&= \sum_{j=0}^{n} |r_j(z)\, q_n(z) - r_n(z)\, q_j(z)|^2 \\
&\geqslant |r_{n-1}(z)\, q_n(z) - r_n(z)\, q_{n-1}(z)|^2,
\end{aligned}$$

the final term being equal to $1/|a_{n-1}|^2$ by (19). Consequently, $(1/[|a_{n-1}|^2 \cdot (1+|q_n|^2)])_{n\geqslant 1}$ is bounded locally uniformly in Ω. Now, if $(a_{n-1})_{n\in\Lambda}$ tends to zero, then for any closed $F\subset\Omega$ there exists an N such that $|q_n(z)|\geqslant 1$ for all $n\in\Lambda$, $n\geqslant N$, and for all $z\in F$. In particular, Ω_0 is asymptotically polefree with respect to Λ, and the local uniform convergence in Ω_0 follows from Theorem 4.1. Note that here we may even establish local uniform convergence in Ω using (14) since

$$\limsup_{n\to\infty,\, n\in\Lambda}\ \max_{z\in F}|\phi(z)-\pi_n(z)|^{1/n} = \limsup_{n\to\infty,\, n\in\Lambda}\ \max_{z\in F}\left|\frac{r_n(z)}{q_n(z)}\right|^{1/n} \leqslant \max_{z\in F}\delta(z)<1. \qquad \blacksquare$$

COROLLARY 4.3. *If* $\operatorname{cap}(\partial\Omega_0)=0$ (*or, equivalently*, $\operatorname{cap}(\sigma(A))=0$, *and thus* $\Omega_0=\Omega$) *then there exists a subsequence of* $(\pi_n)_{n\geqslant 0}$ *converging locally uniformly in* Ω *to* ϕ, *with error decreasing faster than any geometric sequence.*

Proof. By Theorem 2.10(c), $\operatorname{cap}(\partial\Omega_0)=0$ implies that $\kappa_{\sup}=0$ and $g_{\inf}(z)=+\infty$ for all $z\in\Omega$. As a consequence of $\kappa_{\sup}=0$, there is a subsequence of $(a_n)_{n\geqslant 0}$ tending to zero. Thus the assertion follows from Theorem 4.1 together with Corollary 4.2. $\blacksquare$

In the final part of this section, we are concerned with uniform convergence in subsets of the whole resolvent set. As a counterpart of Proposition 3.2(a) and (33), we may prove uniform convergence of quite a dense subsequence in some neighborhood of any $\zeta\in\Omega$ with geometric rate. This generalizes a result of Ambroladze [1, Corollaries 3 and 4], who studied the special case of real recurrence coefficients.

THEOREM 4.4. *For any* $\zeta\in\Omega$ *there exist a neighborhood* $U\subset\Omega$ *of* ζ *and* $(\eta_n)_{n\geqslant 0}\subset\{0,1\}$ *such that*

$$\limsup_{n\to\infty}|\phi(z)-\pi_{n+\eta_n}(z)|^{1/(2n)}\leqslant e^{-g_{\inf}(z)}$$

uniformly for $z\in U$.

Proof. Let U and $(\varepsilon_n)_{n\geqslant 0}$ be chosen as in Lemma 3.3. Then

$$|\phi(z)-\pi_{n+1-\varepsilon_n}(z)|^{1/(2n)}\leqslant|4\cdot a_n\cdot r_n(z)\cdot r_{n+1}(z)|^{1/(2n)}, \qquad n\geqslant 0, \quad z\in U.$$

Thus the assertion follows by combining (15) with Theorem 2.5(c). $\blacksquare$

5. SINGULARITIES OF THE WEYL FUNCTION

In the preceding sections, we have given convergence results for $(\pi_n)_n$ in the resolvent set Ω or its outer component. Its limit, the Weyl function ϕ, is analytic in Ω. In view of the Baker–Gammel–Wills conjecture, it is of some interest to know whether the singularities of (some continuation of) the Weyl function already determine at least partly the shape of the resolvent set (or its outer component).

In the rest of this section it will be useful to consider an additional classification of the spectrum $\sigma(A)$; here we will follow Kato [14, Sect. IV.5.6].

LEMMA 5.1. (a) *If z is an eigenvalue of A then its geometric multiplicity equals one.*

(b) *The (non-empty and compact)* essential spectrum $\sigma_{ess}(A) \subset \sigma(A)$ *consists of $z \in \mathbb{C}$ such that the range of $(zI - A)$ is not closed.*

(c) *Elements of $\sigma(A) \backslash \sigma_{ess}(A)$ are eigenvalues of A.*

(d) *The boundary $\partial\Omega$ of the resolvent set consists of isolated points of $\sigma(A)$ and of a subset of $\partial\sigma_{ess}(A)$.*

Proof. For a proof (a), notice that the eigenspace of an eigenvalue z of A is given by $\{y = (y_j)_{j \geqslant 0} \in \ell^2 : (zI - A)y = 0\}$. A comparison with the recurrence relation for $(q_n(z))_{n \geqslant 0}$ shows that necessarily $y_n = y_0 \cdot q_n(z)$, $n \geqslant 0$, implying (a).

In order to show the other assertions, it is useful to take a slightly more general point of view. Let $\Pi: \ell^2 \to \ell^2$ denote the complex conjugation operator, i.e., $\Pi(y_j)_{j \geqslant 0} = (\overline{y_j})_{j \geqslant 0}$. For a bounded linear operator $T: \ell^2 \to \ell^2$, we denote by $\mathcal{N}(T)$ its hullspace, and by $\mathcal{R}(T)$ its range. Such an operator is called Π-symmetric if its adjoint verifies $T^* = \Pi T \Pi$. For instance, bounded second order difference operators with matrix representation (7) are easily shown to be Π-symmetric. Furthermore, with T also $zI - T$ is Π-symmetric for all $z \in \mathbb{C}$.

Let T be Π-symmetric, and $\mathcal{R}(T)$ be closed. Since $\mathcal{N}(T) = \Pi(\mathcal{N}(T^*))$, we seem from [14, Lemma III.1.40 and Problem III.5.27] that the nullity index $\dim \mathcal{N}(T)$ coincides with the deficiency index, i.e., the codimension of $\mathcal{R}(T)$. It follows from the characterization given in [14, Section IV.5.6] that the essential spectrum of a Π-symmetric operator T is the set of all $z \in \mathbb{C}$ such that $\mathcal{R}(zI - T)$ is not closed, or $\dim \mathcal{N}(zI - T) = \infty$. Thus, assertion (b) is a consequence of assertion (a), and (c) is valid for any Π-symmetric operator. Finally, property (d) is true for any closed operator (see [14, Problem IV.5.37]). ∎

As it becomes clear from the following example, there are no further restrictions for the shape of the (essential) spectrum of second order difference operators.

EXAMPLE 5.2. Let $E \subset \mathbb{C}$ be compact. Furthermore, let $(b_n)_{n\geqslant 0}$ be dense in E, and suppose that for any isolated element e of E there exists an infinite number of indices n with $b_n = e$. We consider the bounded linear operator $\tilde{A}$ with diagonal matrix representation, i.e., $\tilde{b}_n = b_n$ and $\tilde{a}_n = 0$, $n \geqslant 0$. By construction, b_k is an eigenvalue of $\tilde{A}$ for any $k \geqslant 0$, with geometric multiplicity given by the multiplicity of b_k in $(b_n)_{n\geqslant 0}$. From [14, Theorem IV.5.2], we may conclude that the range of $zI - \tilde{A}$ is not closed iff for any $\varepsilon > 0$ there exists an $n \geqslant 0$ with $0 < |b_n - z| < \varepsilon$. Also, with the notations of the proof of Lemma 5.1, the operator $\tilde{A}$ is Π-symmetric, and thus $\sigma_{ess}(\tilde{A}) = E$.

Let $(a_n)_{n\geqslant 0}$ tend to zero. The operator A resulting from (7) is a compact perturbation of $\tilde{A}$, and thus has the same essential spectrum $\sigma_{ess}(A) = E$ by [14, Chap. IV, Theorem 5.35].

Let the complement of E be connected. From Lemma 5.1(d) it follows that the spectrum of A is the set E plus a countable set of isolated points with accumulation points on E. In particular, we have $\Omega = \Omega_0$, and $\mathrm{cap}(\partial\Omega_0) = \mathrm{cap}(E)$.

If the complement of E consists of a finite number of connected components, then by possibly making $\sup_n |a_n|$ smaller, we may insure that there is an element of Ω in any connected component of the complement of E (see, e.g., [14, Theorem IV.5.22]). It follows from [14, Sect. IV.5.6] that $\sigma(A)$ is the set E plus a countable set of isolated points with accumulation points on E. In particular, we have constructed a bounded second order difference operator with resolvent set consisting of several components.

For the particular case of a self-adjoint A it is well-known (see, e.g., [14, Chap. V.3.5]) that any isolated point of $\sigma(A)$ is an eigenvalue of A with geometric and algebraic multiplicity 1; in particular, the corresponding Weyl function has simple poles (namely, the isolated mass points of the spectral measure, see, e.g., [16, Proposition II.8.4]). Also, the remaining part of $\sigma(A)$ coincides with $\sigma_{ess}(A)$. From the Stieltjes–Perron inversion formula one may deduce that the Weyl function of a self-adjoint A does not have neither an analytic continuation in some set larger than $\Omega = \Omega_0$, nor a meromorphic continuation in some set larger than $\mathbb{C} \backslash \sigma_{ess}(A)$.

In the general case, we may at least describe the behavior of ϕ at isolated points of the spectrum.

THEOREM 5.3. *If ζ is an isolated point of $\sigma(A)$, then $\zeta \in \sigma_{ess}(A)$ iff ϕ has an essential singularity in ζ, and ζ is an eigenvalue of algebraic multiplicity $m < \infty$ iff ϕ has a pole of multiplicity m.*

Proof. By assumption, there exists some closed disk U centered at ζ, with $\sigma(A) \cap U = \{\zeta\}$. If ζ is an eigenvalue of A, then we denote by $m' \in \mathbb{N} \cup \{\infty\}$ its algebraic multiplicity (its geometric multiplicity is one by

Lemma 5.1(a)), and otherwise we write $m' = \infty$. Notice that ϕ is analytic in $U \setminus \{\zeta\} \subset \Omega$, i.e., ϕ has an isolated singularity at ζ. If this is a pole, then we denote by m the multiplicity (including $m = 0$ if ϕ has an analytic continuation in U), and otherwise we write $m = \infty$.

It is shown in [14, Theorem IV.5.28] that $z \in \sigma_{ess}(A)$ iff $m' = \infty$. The resolvent R has a Laurent series at ζ, and from the restriction for the geometric multiplicity, we may conclude (see, e.g., [14, Sect. III.6.5]) that the principal part is finite iff $m' < \infty$, more precisely,

$$m' = \inf\{n \geqslant 0: R_\ell = 0 \text{ for all } \ell \geqslant n\}, \qquad R_n := \int_{\partial U} (z-\zeta)^n R(z)\, dz,$$

where the integral has to be taken in the sense of Dunford–Taylor. Notice that R_n is a bounded operator defined on ℓ^2, which is zero iff $(e_j, R_n e_k) = 0$ for all $j, k \geqslant 0$. From (13) we get $(e_0, R_n e_0) = \int_{\partial U} (z-\zeta)^n \phi(z)\, dz$, and thus $m \leqslant m'$. On the other hand, if m is finite, then again by (13) the functions $(z-\zeta)^\ell \cdot (e_j, R(z)\, e_k)$ are analytic in some neighborhood of U for all $\ell \geqslant m$ and for all $j, k \geqslant 0$. This implies that $m' \leqslant m$, and consequently $m' = m$. ▮

Corollary 5.4. *Let D be some bounded domain with $\partial D \subset \Omega(A)$. The Weyl function ϕ has an analytic continuation in $D \cup \Omega(A)$ iff $D \subset \Omega(A)$. Similarly, ϕ has a meromorphic continuation in $D \cup \Omega(A)$ iff $D \cap \sigma_{ess}(A)$ is empty.*

Proof. If $D \subset \Omega$ then the assertion is trivial. Suppose now that $D \cap \sigma_{ess}(A)$ is empty. It follows from Lemma 5.1(d) that D contains only (a finite number of) isolated elements of $\sigma(A)$, and thus ϕ has a meromorphic continuation in D by Theorem 5.3.

In order to show the converse, let ϕ have a meromorphic continuation in $D \cup \Omega(A)$. By assumption, ϕ is analytic in some neighborhood of ∂D, and thus may only have a finite number of poles (counting multiplicities) in D. Let λ by a polynomial of minimal degree such that $\phi \cdot \lambda$ is analytic in D. From (14), we get by applying the maximum principle for analytic functions

$$\max_{z \in D} |\lambda(z)(q_k(z)\, \phi(z) - p_k(z))\, r_k(z) \cdot q_j(z)|$$

$$\leqslant \max_{z \in \partial D} |\lambda(z)| \cdot \max_{z \in \partial D} \beta(z) \cdot \max_{z \in \partial D} \delta(z)^{k-j}$$

for all $0 \leqslant j \leqslant k$, with all quantities on the right being finite, and $\max_{z \in \partial D} \delta(z) < 1$ by Lemma 2.3. It follows from the criterion [2, Theorem 1] for the resolvent set (see also [8, Theorem 2.1]) that $\{z \in D: \lambda(z) \neq 0\}$ is a subset of Ω. Furthermore, zeros of λ may not be elements of the resolvent set since otherwise ϕ would be analytic in a neighborhood of such a point. Also, zero of λ are necessarily isolated points of $\sigma(A)$, and thus elements of $\sigma(A)\backslash\sigma_{ess}(A)$, as shown in Theorem 5.3. ∎

Also, by combining Lemma 5.1(d) and Theorem 5.3, we obtain

COROLLARY 5.5 *Suppose that* $D := \mathbb{C}\backslash\sigma_{ess}(A)$ *is connected. Then the Weyl function* ϕ *is meromorphic in* D, *and* $\sigma(A)\backslash\sigma_{ess}(A)$ *coincides with the set of poles of* ϕ *in* D.

For asymptotically periodic second order difference operators A with matrix representation (7), it is shown in [8, Lemma 2.5] that the essential spectrum has empty interior and connected complement. Thus Corollary 5.5 contains as special case the characterization of [8, Theorem 2.7]. Notice also that for this special case an explicit formula for $\sigma_{ess}(A)$ is given in [8, Remark 2.8].

Let us denote by Δ' the largest open disk centered at infinity in which the Weyl function ϕ has a meromorphic continuation. The *Baker–Gammel–Wills conjecture* [3, 19] says that there is a subsequence of $(\pi_n)_{n\geqslant 0}$ converging to ϕ locally uniformly in the set obtained by dropping from Δ' the poles of ϕ. We denote by Δ the largest open disk centered at infinity which has an empty intersection with $\sigma_{ess}(A)$. From Corollary 5.4, we know that $\Delta \subset \Delta'$, and we conjecture[6] that Δ coincides with Δ'.

Let us mention two interesting implications: By the homographic invariance property of diagonal Padé approximants, the validity of our conjecture would imply that the convex hull of the essential spectrum is the maximal convex set outside of which the Weyl unction has a meromorphic continuation. Furthermore, in view of Theorem 4.1, for proving the Baker–Gammel–Wills conjecture for bounded *J*-fractions it would be sufficient to show that there exists a subsequence of Padé approximants asymptotically having no poles in $\Delta\backslash\sigma(A)$ (for a special case see, e.g., Corollary 4.2).

COROLLARY 5.6. *If* $\sigma_{ess}(A)$ *(or* $\sigma(A)$*) is at most countable then there is local uniform convergence of a subsequence of Padé approximants in the maximal domain of analyticity of the corresponding Weyl function. In particular,* $\Delta = \Delta'$, *and the Baker–Gammel–Wills conjecture is valid.*

Proof. It is shown in [14, Theorem IV.5.33] that if $\sigma_{ess}(A)$ is at most countable then also $\sigma(A)$ is at most countable. In particular, $\operatorname{cap}(\sigma(A)) = 0$, and, by Corollary 4.3, a subsequence of $(\pi_n)_{n\geqslant 0}$ converges locally uniformly in Ω to the Weyl function ϕ. For the first part of the assertion it remains to show ϕ has no analytic continuation in any larger set as Ω.

[6] As seen above, this conjecture is true at least for the case of self-adjoint A.

Notice that any element of a closed and at most countable set $E \subset \mathbb{C}$ is either isolated, or is a limit of a sequence of isolated elements of E. Taking $E = \sigma(A)$, we may conclude from Theorem 5.3 that $z \in \sigma(A) = \mathbb{C} \setminus \Omega$ is either a pole of ϕ, or an essential singularity of ϕ, or a limit point of isolated singularities of ϕ, as claimed above. Moreover, since accumulation points of isolated points of $\sigma(A)$ are elements of $\sigma_{ess}(A)$, we see that the maximal domain of meromorphicity of ϕ is given by $\mathbb{C} \setminus \sigma_{ess}(A)$. This implies $\Delta = \Delta'$, and the validity of the Baker–Gammel–Wills conjecture. ∎

A final example illustrates some of the findings of the last two sections.

EXAMPLE 5.7. It is well known (see, e.g., [15, Appendix, Eq. (3.2.2)]) that the kth convergent of the continued fraction

$$1 + \frac{1|}{|(1/w) - \frac{1}{2}} + \frac{1/(2 \cdot 6)|}{|1/w} + \frac{1/(6 \cdot 10)|}{|1/w} + \frac{1/(10 \cdot 14)|}{|1/w} + \cdots$$

is the Padé approximant at zero of type $[k/k]$ of the function $f(w) = \exp(w)$, and that the sequence of convergents converges to f locally uniformly in $\mathbb{C}$. With $a, b, c \in \mathbb{R}$, $a < b$, we consider the substitution $1/w = (z-a)(z-b)$. One easily verifies that the kth convergent of the continued fraction

$$\frac{z-c|}{|(z-a)(z-b) - \frac{1}{2}} + \frac{1/(2 \cdot 6)|}{|(z-a)(z-b)} + \frac{1/(6 \cdot 10)|}{|(z-a)(z-b)} + \frac{1/(10 \cdot 14)|}{|(z-a)(z-b)} + \cdots$$

equals π_{2k}, the Padé approximant at infinity of order $2k$ of the function $\phi(z) = (z-c) \cdot [\exp(1/[(z-a)(z-b)]) - 1]$. Provided that ϕ has a J-fraction expansion,[7] its even part has the same convergents as the above continued fraction. Comparing coefficients (see, e.g., [15, Theorem II.12]) gives $-a_0^2 + 1/2 = ab - c(a+b-c)$, and

$$b_{2n} = a + b - c, \qquad b_{2n+1} = c,$$

$$-a_{2n}^2 a_{2n+1}^2 = \frac{1}{4(2n+1)(2n+3)},$$

$$-a_{2n+2}^2 - a_{2n+1}^2 = ab - c(a+b-c)$$

for $n \geqslant 0$. We first discuss the case $c \in \{a, b\}$, here $-a_{2n}^2 = 1/(4n+2) = a_{2n-1}^2$ for $n \geqslant 0$. Thus the corresponding operator A is a compact perturbation of

[7] In fact, ϕ has a J-fraction expansion iff the recurrence below has a solution, that is, iff all a_{2n} are different from zero. This is for instance the case if $a \leqslant c \leqslant b$, since then $a_{2n} > c(a+b-c) - ab \geqslant 0$ for all $n \geqslant 0$. Also, in the case $b = -a > 0$ and $c^2 > 1/2 + b^2$, one verifies that $a_{2n} < b^2 - c^2 < -1/2$ for all $n \geqslant 1$.

the operator with diagonal matrix representation $\mathrm{diag}(b_0, b_1, \ldots)$, and $\sigma(A) = \sigma_{ess}(A) = \{a, b\}$ by Corollary 5.5. Here we have local uniform convergence of $(\pi_n)_{n \geqslant 0}$ in Ω according to Corollary 4.2.

Finally, we study the case $c \notin \{a, b\}$. By means of elementary techniques one shows that there exists an $\varepsilon \in \{0, 1\}$ such that $(a_{2n+\varepsilon})_{n \geqslant 0}$ tends to zero, and $(a_{2n-1+\varepsilon})_{n \geqslant 1}$ tends to $c(a+b-c) - ab \neq 0$. Consequently, the corresponding operator A is a compact perturbation of the operator with block diagonal matrix representation $\mathrm{diag}(C, C, C, \ldots)$, where the 2×2 matrix C has the eigenvalues a, b. As above, it follows that $\sigma(A) = \sigma_{ess}(A) = \{a, b\}$. Notice that $\pi_{2n}(c) = 0 = \phi(c)$, and $\pi_{2n+1}(a+b-c) = \infty \neq \phi(a+b-c)$ for all $n \geqslant 0$. Since $c, a+b-c \in \Omega_0$, Corollary 4.2 implies that $\varepsilon = 1$, and consequently $(\pi_{2n})_{n \geqslant 0}$ converges to ϕ locally uniformly in $\Omega = \Omega_0$.

Choosing $0 < b = -a < c$ in the second part of Example 5.7, we see that the spurious pole $a+b-c$ lies outside the convex hull of the spectrum, and is an element of $\Delta = \Delta'$. Thus it seems that Corollary 5.6 may not be essentially improved.

ACKNOWLEDGMENT

The author thanks Professor Valeri Kaliaguine for useful discussions of the results.

REFERENCES

1. A. Ambroladze, On exceptional sets of asymptotic relations for general orthogonal polynomials, *J. Approx. Theory* **82** (1995), 257–273.
2. A. I. Aptekarev, V. A. Kaliaguine, and W. Van Assche, Criterion for the resolvent set of nonsymmetric tridiagonal operators, *Proc. Amer. Math. Soc.* **123** (1995), 2433–2430.
3. G. A. Baker and P. R. Graves-Morris, "Padé Approximants," 2nd ed., Encyclopedia of Mathematics, Cambridge Univ. Press, New York, 1995.
4. D. Barrios, G. López, and E. Torrano, Location of zeros and asymptotics of polynomials defined by three-term recurrence relation with complex coefficients, *Math. Sb.* **184**, No. 11 (1993), 63–92 [In Russian]; English translation, *Russian Acad. Sci. Sb. Math.* **80** (1995), 309–333.
5. D. Barrios, G. López, and E. Torrano, Polynomials generated by asymptotically periodic complex recurrence coefficients, *Math. Sb.* **186** (1995), 3–34 [In Russian]; English translation, *Sb. Math.* **186** (1995), 629–660.
6. D. Barrios, G. López, A. Martínez, and E. Torrano, On the domain of convergence and poles of complex *J*-fractions, *J. Approx. Theory* **93** (1998), 177–200.
7. B. Beckermann, "On the Classification of the Spectrum of Second Order Difference Operators," Publication ANO 379, Université de Lille, 1997; also *Mathematische Nachrichten*, to appear.
8. B. Beckermann and V. A. Kaliaguine, The diagonal of the Padé and the approximation of the Weyl function of second order difference operators, *Constr. Approx.* **13** (1997), 481–510.
9. Yu. M. Berezanskii, Integration of nonlinear difference equation by the inverse spectral problem method, *Dokl. Russian Acad. Nauk.* **281**, No. 1 (1985); English translation, *Soviet Math. Dokl.* **31**, No. 2 (1985), 264–267.

10. S. Demko, W. F. Moss, and P. W. Smith, Decay rates for inverses of band matrices, *Math. Comp.* **43** (1984), 491–499.
11. J. S. Geronimo, E. M. Harrell, and W. Van Assche, On the asymptotic distribution of eigenvalues of banded matrices, *Constr. Approx.* **4** (1988), 403–417.
12. A. A. Gonchar, On uniform convergence of diagonal Padé approximants, *Math. Sb.* **118**, No. 4 (1982), [In Russian]; English translation, *Math. USSR Sb.* **46** (1983), 539–559.
13. V. A. Kaliaguine, On rational approximation of the resolvent function of second order difference operator, *Uspekhi Mat. Nauk.* **49**, No. 3 (1994), 181–182 [In Russian]; English translation, *Russian Math. Surveys* **49**, No. 3 (1994), 187–189.
14. T. Kato, "Perturbation Theory for Linear Operators," Springer-Verlag, New York/Berlin, 1996.
15. L. Lorentzen and H. Waadeland, "Continued Fractions with Applications," North-Holland, Amsterdam, 1992.
16. E. M. Nikishin and V. N. Sorokin, "Rational Approximations and Orthogonality," Nauka, Moscow, 1988; English translation, *Translations of Mathematical Monographs*, Vol. 92, Amer. Math. Soc., Providence, 1991.
17. J. L. Schiff, "Normal Families," Universitext, Springer-Verlag, New York/Berlin, 1993.
18. H. Stahl, On the convergence of generalized Padé approximants, *Constr. Approx.* **5** (1989), 221–240.
19. H. Stahl, Conjectures around the Baker–Gammel–Wills conjecture, *Constr. Approx.* **13** (1997), 287–292.
20. H. Stahl and V. Totik, "General Orthogonal Polynomials," Encyclopedia of Mathematics, Cambridge Univ. Press, New York, 1992.
21. M. Tsuji, "Potential Theory in Modern Function Theory," Dover, New York, 1959.
22. J. L. Ullman, A survey of exterior asymptotic for orthogonal polynomials associated with a finite interval and a study of the case of general weight measures, *in* "Approximation Theory and Spline Functions" (S. P. Singh *et al.*, Eds.), pp. 467–478, Reidel, Dordrecht, 1984.
23. E. A. van Doorn, Representations and bounds for zeros of orthogonal polynomials and eigenvalues of sign-symmetric tri-diagonal matrices, *J. Approx. Theory* **51** (1987), 254–266.
24. H. S. Wall, "Analytic Theory of Continued Fractions," Chelsea, New York, 1973.
25. H. Widom, Polynomials associated with measures in the complex plane, *J. Math. Mech.* **16** (1967), 997–1013.